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Dynamics assignment problem of linear systems

Abstract: - A new dynamic assignment problem for linear standard time – invariant systems is formulated and solved. Necessary and sufficient conditions are established for the existence of the state – output derivative feedbacks such that the closed – loop characteristic polynomial is equal to the desired one of the degree less than the system order. A procedure is derived for computation of gain matrices of the feedbacks.

1. INTRODUCTION

It is well – known [4,5,2] that if the pair of matrices (A, B) of a linear system is controllable then there exists a state – feedback gain matrix K such that $\det[I_n s-(A+BK)]=p(s)$, where $p(s)=s^n+a_{n-1}s^{n-1}+...+a_1 s+a_0$ is a given arbitrary n degree polynomial. By changing K we may modify arbitrarily only the coefficients $a_0, a_1, ..., a_{n-1}$ but we not able to change the degree n of $p(s)$.

In this paper it will be shown that by suitable choice of two gain matrices of state – output derivative feedbacks it is possible to change also the degree of $p(s)$. Necessary and sufficient conditions will be established for the existence of state – output derivative feedbacks such that the closed – loop characteristic polynomial is equal to the desired one of the degree less than the system order. A procedure for computation of gain matrices will be derived and illustrated by a numerical example. Such dynamic assignment problem arises for example in design of the perfect observers for linear standard systems [1,3].

2. PROBLEM STATEMENT

Let $R^{n\times m}$ be the set of $n\times m$ real matrices and $R^n := R^{n\times 1}$. Consider the standard linear continuous – time system

$$\begin{aligned} \dot{x} &= Ax+Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x\in R^n$, $u\in R^m$, $y\in R^p$ are the state input and output vectors, respectively and $A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{p\times n}$, It is assumed that rank B=m, rank C=p and the pair (A, B) is controllable, i.e.

$$rank[Is-A,B]=n \text{ for all } s\in C \text{ (the field of complex numbers)} \tag{2}$$

To the system (2) let us apply the state – output derivative feedback

$$u=v+Kx-F\dot{y} \tag{3}$$

where $v \in R^{m}$ is a new input vector and $K \in R^{m\times n}$, $F \in R^{m\times p}$.

From (1) and (3) we obtain

$$E\dot{x} = (A + BK)x + Bv$$
$$y = Cx \qquad (4)$$

where $E := I_n + BFC$, I_n is the $n\times n$ identity matrix. The dynamic assignment problem can be stated as follows. Given matrices A, B ,C and a desired polynomial

$$p(s) = p_r s^r + \ldots + p_1 s + p_0 \quad (r<n) \qquad (5)$$

find gain matrices K and F of (3) such that the closed – loop characteristic polynomial is equal to the desired polynomial (5), i.e.

$$\det[Es - (A + BK)] = p(s) \qquad (6)$$

We shall establish conditions for the existence of a solution to the problem and we shall give a procedure for computation of the gain matrices K and F of (3).

3. PROBLEM SOLUTION

To simplify the notation we shall consider the solution of the problem for single – input single – output system (m=p=1). It is well – known [4,5,2] that if (2) holds then there exist a nonsingular matrix $P \in R^{n\times n}$ such that

$$\overline{A} = PAP^{-1} = \begin{bmatrix} 0 & I_{n-1} \\ \hline & a \end{bmatrix},\ a = -[\overline{a}_0\ \overline{a}_1 \ldots \overline{a}_{n-1}],\ \overline{B} = PB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ \overline{C} = CP^{-1} = [\overline{c}_0\ \overline{c}_1 \ldots \overline{c}_{n-1}] \qquad (7)$$

Note that

$$\overline{C}\overline{B} = CP^{-1}PB = CB = \overline{c}_{n-1} \qquad (8)$$

and

$$\overline{E} := (I_n + \overline{B}F\overline{C}) = P(I_n + BFC)P^{-1} = PEP^{-1} \qquad (9)$$

Theorem 1. Let the condition (2) be satisfied and the matrices $\overline{A}$, $\overline{B}$ have the form (7). There exists a matrix F such that

$$\overline{E} = I_n + \overline{B}F\overline{C} = \begin{bmatrix} I_{n-1} & 0 \\ \overline{e} & \end{bmatrix},\ \overline{e} = [\overline{e}_0\ \overline{e}_1 \ldots \overline{e}_{n-2}] \qquad (10)$$

$$CB \neq 0 \tag{11}$$

Proof. If (11) holds then for

$$F = [f] = \left[-\frac{1}{CB}\right] = \left[-\frac{1}{\bar{c}_{n-1}}\right] \tag{12}$$

we obtain

$$\bar{E} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \left[-\frac{1}{\bar{c}_{n-1}}\right] [\bar{c}_0\, \bar{c}_1 \ldots \bar{c}_{n-2}] = \left[\begin{array}{c:c} I_{n-1} & 0 \\ \hdashline \bar{e} & \end{array}\right].$$

where $\bar{e} = -\frac{1}{\bar{c}_{n-1}}[\bar{c}_0\, \bar{c}_1 \ldots \bar{c}_{n-2}]$. Note that $\bar{E}$ has the form (10) only if $\bar{c}_{n-1} \neq 0$ and this implies $\overline{CB} \neq 0$ and by (8) $CB \neq 0$.□

Theorem 2. There exist gain matrices K and F satisfying (6) if and only if the conditions (2) and (11) are satisfied.

Proof. By theorem 1 there exists F such that $\bar{E}$ has the form (10) if and only if the conditions (2) and (11) are satisfied. If (2) and (11) hold then using (6), (7) and (9) we may write

$$\det[Es-(A+BK)] = \det\left[P(Es-(A+BK))P^{-1}\right] = \det\left[\bar{E}s-(\bar{A}+\bar{B}\bar{K})\right] \tag{13}$$

where

$$K = \bar{K}P = \left[\bar{k}_1\, \bar{k}_2 \ldots \bar{k}_n\right] P \tag{14}$$

Taking into account (13), (10), (7) and (14) it is easy to check that

$$\begin{aligned} &\det[Es-(A+BK)] = \\ &= \begin{vmatrix} s & -1 & \cdots & 0 & 0 \\ 0 & s & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & \cdots & s & -1 \\ \bar{e}_0 s+\bar{a}_0-\bar{k}_1 & \bar{e}_1 s+\bar{a}_1-\bar{k}_2 & \cdots & \bar{e}_{n-2}s+\bar{a}_{n-2}-\bar{k}_{n-1} & \bar{a}_{n-1}-\bar{k}_n \end{vmatrix} = \\ &= \bar{e}_0 s+\bar{a}_0-\bar{k}_1+\left(\bar{e}_1 s+\bar{a}_1-\bar{k}_2\right)s+\ldots+\left(\bar{e}_{n-2}s+\bar{a}_{n-2}-\bar{k}_{n-1}\right)s^{n-2}+\left(\bar{a}_{n-1}-\bar{k}_n\right)s^{n-1} = \\ &= \bar{a}_0-\bar{k}_1+\left(\bar{e}_0+\bar{a}_1-\bar{k}_2\right)s+\ldots+\left(\bar{e}_{n-3}+\bar{a}_{n-2}-\bar{k}_{n-1}\right)s^{n-2}+\left(\bar{e}_{n-2}+\bar{a}_{n-1}-\bar{k}_n\right)s^{n-1} \end{aligned} \tag{15}$$

Comparison of the right sides of (6) and (15) yields

$$p_0 = \bar{a}_0 - \bar{k}_1,\ p_1 = \bar{e}_0 + \bar{a}_1 - \bar{k}_2,\ \dots,\ p_r = \bar{e}_{r-1} + \bar{a}_r - \bar{k}_{r+1},\ \bar{e}_r + \bar{a}_{r+1} - \bar{k}_{r+2} = 0,\ \dots,\ \bar{e}_{n-2} + \bar{a}_{n-1} - \bar{k}_n = 0 \quad (16)$$

Knowing the coefficients $p_0, p_1, \dots, p_r$, $\bar{e}_0, \bar{e}_1, \dots, \bar{e}_{n-2}$ and $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{n-2}$ from (16) we may find the entries of K

$$\bar{K} = [\bar{a}_0 - p_0, \bar{e}_0 + \bar{a}_1 - p_1, \dots, \bar{e}_{r-1} + \bar{a}_r - p_r, \bar{e}_r + \bar{a}_{r+1}, \dots, \bar{e}_{n-2} + \bar{a}_{n-1}] \quad (17)$$

Using the same arguments as for standard linear systems [1,2,4,5] it is easy to show that the conditions (2) and (11) are also necessary. □

If the conditions (2) and (11) are satisfied then the matrices F and K can be computed by the use of the following procedure.

Procedure

Step 1. Knowing A compute the coefficients $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{n-1}$ of the polynomial

$$\det[Is - A] = \det[Is - \bar{A}] = s^n + \bar{a}_{n-1}s^{n-1} + \dots + \bar{a}_1 s + \bar{a}_0 \quad (18)$$

Step 2. Using (7) find the matrices $\bar{A}$, $\bar{B}$.
Step 3. Compute the matrix

$$P = [\bar{B}, \bar{A}\bar{B}, \dots, \bar{A}^{n-1}\bar{B}][B, AB, \dots, A^{n-1}B]^{-1} \text{ and } \bar{C} = CP^{-1} \quad (19)$$

Step 4. Using (12) find F and $\bar{e} = -\frac{1}{\bar{c}_{n-1}}[\bar{c}_0\, \bar{c}_1 \dots \bar{c}_{n-2}]$.

Step 5. Compute the matrix

$$\bar{K} = [\bar{a}_0 - p_0, \bar{e}_0 + \bar{a}_1 - p_1, \dots, \bar{e}_{r-1} + \bar{a}_r - p_r, \bar{e}_r + \bar{a}_{r+1}, \dots, \bar{e}_{n-2} + \bar{a}_{n-1}] \quad (20)$$

4. EXAMPLE

For system (1) with

$$A = \begin{bmatrix} -2 & -1 & -1 \\ 3 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, C = [2 \ \ 2 \ \ 1] \quad (21)$$

find gain matrices $F = [f]$ and $K = [k_1 \ \ k_2 \ \ k_3]$ such that the closed – loop characteristic polynomial is equal to the one

$$p(s) = 2s + 4 \quad (22)$$

The system satisfied the conditions (2) and (11) since

$$rank[Is - A, B] = rank\begin{bmatrix} s+2 & 1 & 1 & -1 \\ -3 & s-2 & -2 & 1 \\ -2 & -2 & s & 1 \end{bmatrix} = 3 \text{ for all } s \in C$$

and

$$CB = [2 \quad 2 \quad 1]\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 1$$

Using the procedure we obtain

Step 1.

$$\det[Is - A] = \begin{bmatrix} s+2 & 1 & 1 \\ -3 & s-2 & -2 \\ -2 & -2 & s \end{bmatrix} = s^3 - 3s - 2 \text{ and } \bar{a}_0 = -2,\ \bar{a}_1 = -3,\ \bar{a}_2 = 0.$$

Step 2. The matrices $\overline{A}$, $\overline{B}$ have the form

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix},\ \overline{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{23}$$

Step 3. Using (19), (21) and (23) we compute

$$P = [\overline{B}, \overline{A}\overline{B}, \overline{A}^2\overline{B}][B, AB, A^2B]^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

and

$$\overline{C} = CP^{-1} = [2 \quad 2 \quad 1]\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = [1 \quad 2 \quad 1]$$

Step 4. From (12) we have $F = \left[-\frac{1}{\bar{c}_2}\right] = [-1]$ and $\bar{e} = [\bar{e}_0 \quad \bar{e}_1] = -\frac{1}{\bar{c}_2}[\bar{c}_0 \quad \bar{c}_1] = [-1 \quad -2]$.

Step 5. Using (20) and taking into account (22) we obtain

$$K=[\bar{a}_0-p_0,\bar{e}_0+\bar{a}_1-p_1,\bar{e}_1+\bar{a}_2]P=[-6 \quad -6 \quad -2]\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 1 & 0 & 2\end{bmatrix}=[-8 \quad -6 \quad -4]$$

5. CONCLUDING REMARKS.

Necessary and sufficient conditions (2) and (11) have been established for the existence of the state – output feedback (3) such that the equality (6) is satisfied for any given polynomial $p(s)$ of the degree r<n. A procedure have been derived for computation of the gain matrices F and K of (3) and illustrated by numerical example. By slight modifications the presented method can be extended for multi – input multi – output linear continuous – time and discrete – time systems. An extension of this method for linear two – dimensional systems [2] will be considered in a separate paper.

6. REFERENCES

[1] L. Dai, *Singular Control Systems*, Springer Verlag, Berlin – Tokyo 1989.

[2] T. Kaczorek, *Linear Control Systems*, vol. 1 and 2, Research Studies Press and J. Wiley, New York 1993.

[3] T. Kaczorek, *Perfect Observers for Singular 2D Linear Systems,* Bull. Pol. Acad. Techn. Sci. 2000.

[4] T. Kailath, *Linear Systems*, EnglewoodCliffs: Prentice Hall, 1980.

[5] W. M. Wonham, *Linear Multivariable Control: a Geometric Approach*, Springer, New York 1979.