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INFLUENCE OF THE STATE-FEEDBACK ON CYCLICITY OF LINEAR SYSTEMS

Abstract. *It is shown that every second order nonzero minor of the polynomial matrix* $P_A\left([I_n s - A]^{-1} = \frac{P_A}{d}, A \in R^{n\times n}, n \geq 2\right)$ *is divisible (with zero remainder) by the polynomial* d *if and only if the characteristic polynomial* $\varphi(s) = \det[I_n s - A]$ *is equal to the minimal polynomial* $\Psi(s)$ *of* A. *If the transfer matrix of m-inputs and p-outputs* $\min(m,p) \geq 2$ *linear system is written in the standard form* $T = \frac{P}{d}$ *(d is the minimal common denominator), then every second order nonzero minor of P is divisible by d if and only if* $q = d$, *where q is the McMillan polynomial of T. If the pair (A,b) of single-input system is controllable then the closed-loop matrix* $A_c = A + bk$ *(k is a gain matrix) is cyclic if and only if the matrix A is also cyclic. If the pair (A,B) of m-input system is controllable and A is not cyclic then there exists a feedback gain matrix K such that* $A_c = A + BK$ *is cyclic. If the pair (A,B) is uncontrollable and A is not cyclic then there exists a feedback gain matrix K such that* $A_c = A + BK$ *is cyclic if and only if the submatrix* A_3 *of (50) of the uncontrollable part of the system is cyclic.*

Streszczenie. *W pracy wykazano, że każdy niezerowy minor stopnia drugiego macierzy* $P_A\left([I_n s - A]^{-1} = \frac{P_A}{d}, A \in R^{n\times n}, n \geq 2\right)$ *jest podzielny bez reszty przez wielomian d wtedy i tylko wtedy, gdy wielomian charakterystyczny* $\varphi(s) = \det[I_n s - A]$ *jest równy wielomianowi minimalnemu* $\Psi(s)$ *macierzy A. Jeżeli macierz transmitancji układu o m-wejściach i p-wyjściach* $\min(m,p) \geq 2$ *jest w postaci* $T = \frac{P}{d}$ *(d jest najmniejszym wspólnym mianownikiem), to każdy niezerowy minor stopnia drugiego macierzy P jest podzielny bez reszty przez d wtedy i tylko wtedy, gdy q=d, przy czym q jest wielomianem McMillana macierzy T. Jeżeli para (A,b) układu o jednym wejściu jest sterowalna, to macierz układu zamkniętego* $A_c = A + BK$ *(k jest wektorem wzmocnień)jest cykliczna wtedy i tylko wtedy, gdy macierz A jest również cykliczna. Jeżeli para (A,B) układu o m-wejściach jest sterowalna i macierz A nie jest cykliczna, to istnieje macierz sprzężeń zwrotnych K taka, że macierz* $A_c = A + BK$ *jest cykliczna. Jeżeli para (A,B) jest niesterowalna i macierz A jest niecykliczna, to istnieje macierz sprzężeń zwrotnych K taka, że macierz* $A_c = A + BK$ *jest*

cykliczna wtedy i tylko wtedy, gdy podmacierz A_3 (w (50)) części niesterowalnej układu jest cykliczna.

1. INTRODUCTION

In the monograph [5] Rosenwasser and Lampe have introduced the notion of the simple matrix (einfache Matrix, which is equivalent to the cyclic matrix [4]) and the notion of the normal matrix (normale Matrix). They have shown that if the normal transfer matrix is written in the standard form $T = \frac{P}{d}$ (d is the minimal common denominator), then every second order nonzero minor of P is divisible by d with zero remainder. Some implications of this approach to electrical circuits have been discussed in [2].
In this paper it will be shown that every second order nonzero minor of the polynomial matrix $P_A\left([I_n s - A]^{-1} = \frac{P_A}{d}, A \in R^{n \times n}, n \geq 2\right)$ is divisible (with zero remainder) by the polynomial d if and only if the characteristic polynomial $\varphi(s) = \det[I_n s - A]$ is equal to the minimal polynomial $\Psi(s)$ of A. If the transfer matrix of m-inputs and p-outputs ($\min(m,p) \geq 2$) linear system is written in the standard form $T = \frac{P}{d}$ (d is the minimal common denominator), then every second order nonzero minor of P is divisible by d if and only if $q = d$, where q is the McMillan polynomial of T. If the pair (A,b) of single input system is controllable then the closed-loop matrix $A_c = A + bk$ (k is a gain matrix) is cyclic if and only if the matrix A is also cyclic. If the pair (A,B) of m-input system is controllable and A is not cyclic then there exists a feedback gain matrix K such that $A_c = A + BK$ is cyclic. If the pair (A,B) is uncontrollable and A is not cyclic then there exists a feedback gain matrix K such that $A_c = A + BK$ is cyclic if and only if the submatrix A_3 of (50) of the uncontrollable part of the system is cyclic.

2. PRELIMINARIES

Let $R^{m \times n}$ be the set of $m \times n$ real matrices and $R^n := R^{n \times 1}$.
Consider the linear continuous-time system

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

where $x = x(t) \in R^n$ is the state vector, $u = u(t) \in R^m$ and $y = y(t) \in R^p$ are the input and output vectors, respectively and $A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}, D \in R^{p \times m}$.
The transfer matrix of the system (1) is given by

$$T(s) = C[I_n s - A]^{-1} B + D \tag{2}$$

which can be written in the standard form

$$T(s) = \frac{P(s)}{d(s)} \tag{3}$$

where $P \in R^{p\times m}[s], R^{p\times m}[s]$ is the set of $p\times m$ polynomial matrices and $d(s)$ is the minimal common denominator of all entries of $T(s)$.

In what follows the following elementary row or column operations will be used [1,3]:

1. Multiplication of any row (column) by any nonzero number (scalar).
2. Addition of any row (column) multiplied by a polynomial to an other row (column).
3. Interchange of any rows (columns).

Using elementary row and column operations we may transform any polynomial matrix $P \in R^{p\times m}[s]$ to its Smith canonical form [3,4]

$$P_S(s) = diag[i_1(s), i_2(s), ..., i_r(s), 0, ..., 0] \in R^{p\times m}[s] \tag{4}$$

where $i_1(s),...,i_r(s)$ are monic invariant polynomials satisfying the divisibility condition $i_{k+1}(s) \mid i_k(s)$, i.e. $i_{k+1}(s)$ is divisible with zero remainder by $i_k(s), k = 1,...,r-1$ and $r = rank\, P(s)$.

The invariant polynomials can be determined by the relation

$$i_k(s) = \frac{D_k(s)}{D_{k-1}(s)} \quad (D_0(s) = 1),\ k = 1,...,r \tag{5}$$

where $D_k(s)$ is the greatest common divisor of all the $k\times k$ minors of $P(s)$.

The characteristic polynomial $\varphi(s) = \det[I_n - A]$ of the matrix $A \in R^{n\times n}$ and its minimal polynomial $\Psi(s)$ are related by [1]

$$\Psi(s) = \frac{\varphi(s)}{D_{n-1}(s)} \tag{6}$$

From (4)-(6) it follows that $\Psi(s) = \varphi(s)$ if and only if

$$D_1(s) = D_2(s) = \cdots = D_{n-1}(s) = 1 \tag{7}$$

A matrix $A \in R^{n\times n}$ satisfying (7) (or equivalently $\Psi(s) = \varphi(s)$) is called cyclic (or normal [5]).

3. DIVISIBILITY OF SECOND ORDER MINORS OF CYCLIC MATRICES

For any $A \in R^{n\times n}$ the inverse matrix $[Is - A]^{-1}$ can be written in the form

$$[Is - A]^{-1} = \frac{P_A}{d} \tag{8}$$

where $P_A = P_A(s) \in R^{n\times n}[s]$ and $d = d(s)$ is the minimal common denominator.

Theorem 1. Let $A \in R^{n\times n}$ and $n \geq 2$. Then every second order nonzero minor of P_A is divisible (with zero remainder) by d if and only if the characteristic polynomial $\varphi(s) = \det[Is - A]$ is equal to the minimal polynomial of A, i.e. $\varphi(s) = \Psi(s)$.

Proof. It is well known that any square matrix is similar to its Jordan canonical form, that is, there exists a non-singular matrix $T \in R^{n\times n}$ such that

$$J_A = TAT^{-1} = diag[J_{11}(s_1),\dots,J_{m_1k_1}(s_1),J_{21}(s_2),\dots,J_{m_2k_2}(s_2),\dots,J_{m_qk_q}(s_q)] \tag{9a}$$

where

$$J_m(s) = \begin{bmatrix} s & 1 & 0 & \cdots & 0 & 0 \\ 0 & s & 1 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & 0 & \cdots & s & 1 \\ 0 & 0 & 0 & \cdots & 0 & s \end{bmatrix} \in R^{m\times m}[s] \tag{9b}$$

$s_1, s_2, \dots, s_q$ are the distinct eigevalues of A and the multiplicity of s_j is $m_{j1} + m_{j2} + \dots + m_{jk_j} = m_j$, $j = 1,\dots,q$.

From (9a) it follows that

$$\det[I_n s - J_A] = \det[I_n s - A] = \varphi(s) \tag{10}$$

If (7) holds then

$$J_A = diag[J_{m_1}(s_1), J_{m_2}(s_2),\dots,J_{m_q}(s_q)] \qquad (m_1 + m_2 + \dots + m_q = n) \tag{11}$$

and from (9a) we obtain

$$[I_n s - A]^{-1} = [I_n s - T^{-1}J_A T]^{-1} = T^{-1}[I_n s - J_A]^{-1}T =$$

$$= diag\{[I_{m_1}s - J_{m_1}(s_1)]^{-1}, [I_{m_2}s - J_{m_2}(s_2)]^{-1}, \dots, [I_{m_q}s - J_{m_q}(s_q)]^{-1}\} =$$

$$= diag\left\{\frac{adj[I_{m_1}s - J_{m_1}(s_1)]}{d_1}, \frac{adj[I_{m_2}s - J_{m_2}(s_2)]}{d_2}, \dots, \frac{adj[I_{m_q}s - J_{m_q}(s_q)]}{d_q}\right\} \tag{12}$$

where $d_j = (s - s_j)^{m_j}$, $j = 1,\dots,q$ and $d = d_1 d_2 \cdots d_q$.

From (12) it follows that it is enough to show that every second order nonzero minor of the adjoint matrix $adj[I_{m_j}s - J_{m_j}(s_j)]$ is divisible by d_j for $j = 1,\dots,q$.

Taking into account that

$$adj[I_{m_j}s - J_{m_j}(s_j)] = adj\begin{bmatrix} s-s_j & -1 & 0 & \cdots & 0 & 0 \\ 0 & s-s_j & -1 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & 0 & \cdots & s-s_j & -1 \\ 0 & 0 & 0 & \cdots & 0 & s-s_j \end{bmatrix} =$$

$$= \begin{bmatrix} (s-s_j)^{m_j-1} & (s-s_j)^{m_j-2} & (s-s_j)^{m_j-3} & \cdots & s-s_j & 1 \\ 0 & (s-s_j)^{m_j-1} & (s-s_j)^{m_j-2} & \cdots & (s-s_j)^2 & s-s_j \\ \hdashline 0 & 0 & 0 & \cdots & (s-s_j)^{m_j-1} & (s-s_j)^{m_j-2} \\ 0 & 0 & 0 & \cdots & 0 & (s-s_j)^{m_j-1} \end{bmatrix} \tag{13}$$

it is easy to check that every nonzero second order minor of (13) is divisible by $d_j = (s - s_j)^{m_j}$, $j = 1,\dots,q$.

If $\varphi(s) \neq \Psi(s)$ then for at least one eigenvalue, let say s_j, we have (for $m_{j_1} > m_{j_2}$)

$$J(s_j) = diag[J_{m_{j1}}(s_j), J_{m_{j2}}(s_j)]$$

and

$$[Is - J(s_j)]^{-1} = diag\{[I_{m_{j1}}s - J_{m_{j1}}(s_j)]^{-1}, [I_{m_{j1}}s - J_{m_{j2}}(s_j)]^{-1}\} =$$
$$= \frac{1}{d_{m_{j1}}} diag\{adj[I_{m_{j1}}s - J_{m_{j1}}(s_j)], (s - s_j)^{m_{j1}-m_{j2}} adj[I_{m_{j2}}s - J_{m_{j2}}(s_j)]\} \tag{14}$$

where $d_{m_{j1}} = (s - s_j)^{m_{j1}}$ and the adjont matrices $adj[I_{m_{j1}}s - J_{m_{j1}}(s_j)]$, $adj[I_{m_{j2}}s - J_{m_{j2}}(s_j)]$ are defined in the same way as (13).

It is easy to check that for example the second order minor $\begin{vmatrix} 1 & 0 \\ (s-s_j)^{m_{j2}} & (s-s_j)^{m_{j2}-1} \end{vmatrix}$ of the matrix

$$diag\{adj[I_{m_{j1}}s - J_{m_{j1}}(s_j)], (s - s_j)^{m_{j1}-m_{j2}} adj[I_{m_{j2}}s - J_{m_{j2}}(s_j)]\}$$

is not divisible by $d_{m_{j1}}$.□

Remark 1. Any nondiagonal matrix $A = [a_{ij}] \in R^{n \times n}$ for $n = 2$ is cyclic since $D_{n-1}(s)$ of $[I_n s - A]$ is for $a_{12} \neq 0$ or $a_{21} \neq 0$ a nonzero scalar.

Theorem 2. A matrix $A = [a_{ij}] \in R^{n \times n}$ is cyclic if

$$a_{ij} \begin{cases} = 0 & \text{for} \quad j > i+1 \\ \neq 0 & \text{for} \quad j = i+1 \end{cases} \qquad i, j = 1, \ldots, n \tag{15a}$$

or

$$a_{ij} \begin{cases} = 0 & \text{for} \quad i > j+1 \\ \neq 0 & \text{for} \quad i = j+1 \end{cases} \qquad i, j = 1, \ldots, n \tag{15b}$$

Proof. If (15a) holds then the minor M_{n1} obtained by deleting of the first column and the n-th row of the matrix $[I_n s - A]$ is equal to $M_{n1} = a_{12}a_{23} \cdots a_{n-1,n} \neq 0$. Hence $D_{n-1}(s) = 1$. In this case from (6) we obtain $\varphi(s) = \Psi(s)$. The proof for (15b) is similar (dual). □

In particular case from Theorem 2 it follows that the Frobenius matrix

$$A_F = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{array}\right] \text{ or } A_F^T = \left[\begin{array}{ccccc} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \hline 0 & 0 & \cdots & 1 & a_{n-1} \end{array}\right] \tag{16}$$

is cyclic [1].

4. DIVISIBILITY OF SECOND ORDER MINORS OF TRANSFER MATRICES

The transfer matrix (2) of (1) can be always written in the form (3). If the pair (*A*, *B*) is reachable (controllable) and the pair (*A*, *C*) is observable then

$$P(s) = C\,adj[I_n s - A]B + Dd \text{ and } d = \det[I_n s - A] \tag{17}$$

If $m \geq p$ $(p \geq m)$ and $rank\, C = p$ $(rank B = m)$ then $r = rank\, P(s) = p(m)$ and the Smith canonical form (4) of $P(s)$ is equal to

$$P_S(s) = UP(s)V = \begin{bmatrix} i_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & i_2(s) & \cdots & 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & i_p(s) & 0 & \cdots & 0 \end{bmatrix} \in R^{p\times m}[s] \tag{18}$$

where $U = U(s) \in R^{p\times p}[s], V = V(s) \in R^{m\times m}[s]$ are unimodular matrices of elementary row and column operations, respectively.

From (18) and (3) we have the McMillan canonical form of $T(s)$ [3,4]

$$T_M(s) = \frac{P_S(s)}{d(s)} = \frac{UP(s)V}{d(s)} = \begin{bmatrix} \frac{n_1}{q_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{n_2}{q_2} & \cdots & 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & \frac{n_p}{q_p} & 0 & \cdots & 0 \end{bmatrix} \tag{19}$$

where $\frac{i_k(s)}{d(s)} = \frac{n_k(s)}{q_k(s)}$ for $k = 1,\dots,p$ $(n_1 = i_1, q_1 = d), n_k = n_k(s)$ and $q_k(s)$ are factor coprime polynomials such that $n_k \mid n_{k+1}$ and $q_{k+1} \mid q_k$, $k = 1,\dots,p-1$.

The polynomial

$$q(s) = q_1 q_2 \dots q_p \tag{20}$$

is called the McMilan polynomial of $T(s)$.

From (18) – (20) it follows that $\deg q(s) \geq \deg d(s)$ and

$$q(s) = d(s) \text{ if and only if } q_k(s) = 1 \text{ for } k = 2,\dots,p \quad (q_1(s) = d(s)) \tag{21}$$

In the proof of the following theorem the Binet – Cauchy lemma will be used [1].

Lemma. Let $C = AB$, where $A \in R^{m\times n}, B \in R^{n\times p}$. Then the minor of the q order ($q \leq \min(m,p)$ of the matrix C is given by the formula

$$C^{i_1 i_2 \dots i_q}_{j_1 j_2 \dots j_q} = \sum_{1\leq k_1 < \dots < k_q \leq n} A^{i_1 i_2 \dots i_q}_{k_1 k_2 \dots k_q} B^{k_1 k_2 \dots k_q}_{j_1 j_2 \dots j_q} \tag{22}$$

where $A^{i_1 i_2 \dots i_q}_{k_1 k_2 \dots k_q}$ is the minor consisting of rows $i_1, i_2, \dots, i_q$ and columns $j_1, j_2, \dots, j_q$ of the matrix A. The minors $B^{k_1 k_2 \dots k_q}_{j_1 j_2 \dots j_q}$ and $C^{i_1 i_2 \dots i_q}_{j_1 j_2 \dots j_q}$ are defined in the same way.

Theorem 3. Let $\min(m,p) \geq 2$ and let $T(s)$ be given in the form (3). Then every second order nonzero minor of the polynomial matrix $P(s)$ is divisible (with zero remainder) by $d(s)$ if and only if $q(s) = d(s)$.

Proof. Sufficiency. If $q(s) = d(s)$ then by (21) $q_k(s) = 1$ for $k = 2,\dots,p$ and (19) takes the form

$$T_M(s) = \frac{P_M(s)}{d(s)} \tag{23}$$

and

$$T(s) = U^{-1}(s)T_M(s)V^{-1}(s) = \frac{P(s)}{d(s)}$$

where

$$P_M(s) = \begin{bmatrix} i_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & i_1(s)t_2(s)d(s) & \cdots & 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & i_1(s)t_p(s)d(s) & 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

and

$$P(s) = U^{-1}(s)P_M(s)V^{-1}(s)$$

$U^{-1}(s)$ and $V^{-1}(s)$ are unimodular matrices and some of the polynomials $t_k(s)$, $k = 2,\dots,p$ may be equal to 1.
It is easy to see that every second order nonzero minor of $P_M(s)$ is divisible by $d(s)$. Applying Lemma to the matrix $P(s) = U^{-1}(s)P_M(s)V^{-1}(s)$ we conclude that every nonzero second order minor of $P(s)$ is divisible by $d(s)$.

Necessity. If every nonzero second order minor of $P(s)$ is divisible by $d(s)$ then by Lemma every second order nonzero minor of $P_M(s)$ is also divisible by $d(s)$ since $U^{-1}(s)$ and $V^{-1}(s)$ are unimodular matrices. This implies that the matrix $P_M(s)$ has the form (24) and by (23) we obtain $q_k(s) = 1$ for $k = 2,\dots,p$. In this case from (21) we have $q(s) = d(s)$. □

Remark 2. The Theorem 3 can be also proved by the use of Theorem 1 and Lemma.

Example 1. Consider the transfer matrix

$$T(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} \tag{25}$$

In this case

$$d(s) = (s+1)(s+2)$$

and

$$P(s) = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} \tag{26}$$

The canonical Smith form of (26) is equal to

$$P_S(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+1)(s+2) \end{bmatrix} \tag{27}$$

and the canonical McMillan form

$$T_M(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & 1 \end{bmatrix}$$

Hence

$$q(s) = (s+1)(s+2) = d(s)$$

It is easy to see that $\det P(s) = d(s)$ is divisible by $d(s)$.

Example 2. Consider the transfer matrix

$$T(s)=\begin{bmatrix}\frac{s+2}{(s+1)^2} & 0\\ 0 & \frac{1}{s+1}\end{bmatrix}=\frac{1}{(s+1)^2}\begin{bmatrix}s+2 & 0\\ 0 & s+1\end{bmatrix} \tag{28}$$

In this case $d(s)=(s+1)^2$ and $P(s)$, $P_S(s)$ are given by (26) and (27), respectively. Thus the canonical McMillan form is equal to

$$T_M(s)=\begin{bmatrix}\frac{1}{(s+1)^2} & 0\\ 0 & \frac{s+2}{s+1}\end{bmatrix}$$

Hence $q(s)=(s+1)^3\neq d(s)$.
It is easy to see that $\det P(s)=(s+1)(s+2)$ is not divisible by $d(s)=(s+1)^2$.

5. SYSTEMS WITH STATE-FEEDBACKS.

Let us consider the system (1) with the state-feedback

$$u=v-Kx \tag{29}$$

where $v\in R^m$ is the new input vector and $K\in R^{m\times n}$.
Substitution of (29) into (10) yields

$$\dot{x}=A_c x+Bv \tag{30}$$

where

$$A_c=A+BK \tag{31}$$

5.1. Single-input systems

Consider the single input (m=1) system (1) with (29), $B=b$ and $K=k\in R^{1\times n}$.

Theorem 4. The pair (A,b) is controllable only if the characteristic polynomial $\varphi(s)=\det[Is-A]$ is equal to the minimal polynomial $\Psi(s)$ of A, i.e. $\varphi(s)=\Psi(s)$.

Proof. It is well-known [3,4] that the pair (A,b) is controllable if and only if

$$rank[b,Ab,...,A^{n-1}b]=n \tag{31}$$

If $\Psi(s)\neq\varphi(s)$ then from (6) we have $\deg\Psi(s)=n_1<n$.
Let $\Psi(s)=s^{n_1}+a_{n_1-1}s^{n_1-1}+\cdots+a_1 s+a_0$. Then $A^{n_1}=-a_{n_1-1}A^{n_1-1}-\cdots-a_1A-a_0I_n$ and all columns $A^{n_1}b,...,A^{n-1}b$ in the matrix $[b,Ab,...,A^{n-1}b]$ are linearly dependent on $b,Ab,...,A^{n_1-1}b$.
Therefore, the condition (31) can be satisfied only if $\Psi(s)=\varphi(s)$. □
It is also well-known [3,4] that the pair (A_c,b) is controllable if and only if the pair (A,b) is controllable.

Theorem 5. Let the pair (A,b) be controllable. Then the matrix A_c of the closed-loop system (30) is cyclic if and only if the matrix A of (1) is cyclic.

Proof. Necessity. If the pair (A,b) is controllable then the pair (A_c,b) is also controllable for any feedback gain matrix k. By Theorem 4 the controllability of the pair (A_c,b) implies that A_c is cyclic.

Sufficiency. If the pair (A,b) is controllable then there exists a non-singular matrix T such that [3,4]

$$\overline{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad \overline{b} = Tb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{32}$$

The matrix $\overline{A}$ is cyclic with $\Psi(s) = \det[Is - \overline{A}] = \det[Is - A]$.
Using (32) we may write

$$A_c = A + bk = T^{-1}(\overline{A} + \overline{b}\overline{k})T \tag{33}$$

where

$$\overline{k} = kT^{-1} = [\overline{k}_1, \overline{k}_2, \ldots, \overline{k}_n] \tag{34}$$

Hence the matrix

$$\overline{A}_c = \overline{A} + \overline{b}\overline{k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \overline{k}_1 - a_0 & \overline{k}_2 - a_1 & \overline{k}_3 - a_2 & \cdots & \overline{k}_n - a_{n-1} \end{bmatrix} \tag{35}$$

is cyclic. From (33) it follows that $\det[Is - A_c] = \det[Is - \overline{A}_c]$ and A_c is also cyclic. □
Therefore, we have the following corollary

Corollary 1. If the pair (A,b) is controllable the cyclicity of the matrix A is invariant under the state-feedback.
If the pair (A,b) is not controllable and A is not cyclic then as shows the following example it is possible to choose the feedback gain matrix so $A_c = A + bk$ is cyclic.

Example 1. The pair

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{36}$$

is not controllable and A is not cyclic since $\varphi(s) = \begin{vmatrix} s-1 & 0 \\ 0 & s-1 \end{vmatrix} = (s-1)^2$ and $\Psi(s) = s-1$. It is easy to verify that for $k = [0\ 1]$ the closed-loop matrix $A_c = A + bk = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is cyclic $\Psi(s) = \varphi(s) = (s+1)(s+2)$.
In general case when the single-input system (1) is not controllable there exists a non-singular matrix T such that [3,4]

$$\overline{A} = TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \overline{b} = Tb = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}, \begin{matrix} A_1 \in R^{r\times r}, b_1 \in R^r \\ A_3 \in R^{(n-r)\times(n-r)} \end{matrix} \tag{37}$$

where the pair (A_1, b_1) is controllable and it has the canonical form (32) and $r = rank[b, Ab,..., A^{n-1}b] < n$.

Theorem 6. Let the pair (A, b) be uncontrollable and A be not cyclic. Then there exists a feedback gain matrix k such that $A_c = A + bk$ is cyclic if and only if the matrix A_3 is cyclic.

Proof. Sufficiency. If A_3 is cyclic, A_1 has the Frobenius form and A is not cyclic then the minimal polynomials $\Psi_1(s)$ and $\Psi_3(s)$ of A_1 and A_3 have at least one common factor. The pair (A_1, b_1) is controllable. Thus it is possible to choose k so that the matrix $A_1 + b_1 k$ has a minimal polynomial which has no common factors with $\Psi_3(s)$. In this case the matrix $\overline{A}(A)$ is cyclic.

Necessity. Follows immediately from the fact that $\overline{A}(A)$ is cyclic only if A_3 is cyclic. □

Example 2. Consider the single-input system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -4 & -8 & -5 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

It is easy to check that the pair is not controllable and it has already the desired form (37) with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix}. A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{39}$$

The matrices A_1 and A_3 are cyclic but their minimal polynomials $\Psi_1(s) = \det[Is - A_1] = (s+1)(s+2)^2, \Psi_3(s) = \det[Is - A_3] = (s+1)^2$ have common factor $(s+1)$. Therefore, the matrix A is not cyclic.

The conditions of Theorem 6 are satisfied and there exists a feedback gain matrix $k = [k_1\ k_2\ k_3\ k_4\ k_5]$ such that $A_c = A + bk$ is cyclic. The gain matrix k should be chosen so that the minimal polynomial of $A_{c1} = A_1 + b_1\overline{k}, \overline{k} = [k_1\ k_2\ k_3]$ has no common factors with $\Psi_3(s)$. Let the desired minimal polynomial of A_{c1} be $\Psi_{c1}(s) = (s+2)^3$.

Then

$$A_{c1} = A_1 + b_1\overline{k} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[-4, -4, -1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix}$$

$$k = [\overline{k}\ k_4\ k_5] = [-4, -4, -1, 0, 1]$$

and

$$A_c = A + bk = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -8 & -12 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix} \tag{40}$$

It is easy to check that the matrix (40) is cyclic with the minimal polynomial $\Psi_c(s) = (s+2)^3(s+1)^2$.

4.2 Multi-input systems

Consider the m-inputs system (1) with (29).
If the pair (A,b) is controllable then there exists a non-singular matrix T such that [3,4]

$$\overline{A} = TAT^{-1} = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}, \overline{B} = TB = \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix}, A_{ij} \in R^{d_i \times d_j}, B_i \in R^{d_i \times m} \tag{41a}$$

where

$$A_{ij} = \begin{cases} \begin{bmatrix} 0 & I_{d_i - 1} \\ \hline & -a_i \end{bmatrix} & for \quad i = j \\ \begin{bmatrix} 0 \\ \hline -a_{ij} \end{bmatrix} & for \quad i \neq j \end{cases}, B_i = \begin{bmatrix} 0 \\ \cdots \\ b_i \end{bmatrix} \quad \begin{matrix} a_{ij} = [a_0^{ij} \; a_1^{ij} \ldots a_{d_j - 1}^{ij}] \\ b_i = [0 \cdots 0 \; 1 \; b_{i,i+1} \cdots b_{im}] \end{matrix} \tag{41b}$$

and $d_1, \ldots, d_m$ are the controllability indexes satisfying $\sum_{i=1}^{m} d_i = n$.

Theorem 7. Let A be not cyclic. Then there exists a feedback gain matrix K such that $A_c = A + BK$ is cyclic if the pair (A, B) is controllable.

Proof. If the pair (A,B) is controllable then the pair can be transformed to its canonical form (41).
Let

$$\hat{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}^{-1} = \begin{bmatrix} 1 & b_{12} & \cdots & b_{1m} \\ 0 & 1 & \cdots & b_{2m} \\ \hline 0 & 0 & \cdots & 1 \end{bmatrix}^{-1} \tag{42}$$

then

$$\widetilde{B} = \overline{B}\hat{B} = diag[\widetilde{b}_1, \ldots, \widetilde{b}_m], \widetilde{b}_i = [0 \cdots 0 \; 1]^T \in R^{d_i} \tag{43}$$

Define

$$\overline{K} = \hat{B}^{-1}KT^{-1} = \begin{bmatrix} -a_{n_1} + e_{n_1 + 1} \\ \hline -a_{n_{m-1}} + e_{n_{m-1}+1} \\ -a_{n_m} - d \end{bmatrix} \tag{44}$$

where $n_i = \sum_{k=1}^{i} d_k$, a_{n_i} is the n_i-th row of $\overline{A}, e_i$ is the i-th row of I_n and

$$d = [d_0, d_1, ..., d_{n-1}] \tag{45}$$

Using (42)-(45) it is easy to verify that

$$A_c = T(A+BK)T^{-1} = \overline{A} + \overline{B}KT^{-1} = \overline{A} + \overline{B}\hat{B}\hat{B}^{-1}KT^{-1} = \overline{A} + \widetilde{B}\overline{K} =$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{n-1} \end{bmatrix} \tag{46}$$

The matrix (46) is cyclic.
The desired feedback gain matrix is given by the formula

$$K = \hat{B}\overline{K}T \tag{47}$$

which follows from (44). □

Remark 3. Note that for different (45) we obtain different matrices (46). Hence there exist many gain matrices K solving the problem.

Example 3. Consider the system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -8 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{48}$$

The pair (48) is controllable and has already the form (41) but the matrix A is not cyclic. In this case $d_1 = 2, d_2 = 3, n_1 = d_1, n_2 = d_1 + d_2 = 5, T = I_5, \overline{A} = A$ and $\overline{B} = B$.
To find a feedback gain matrix $K = [k_{ij}] \in R^{2\times 5}$ such that $A_c = A + BK$ is cyclic we compute using (42), (43), (44) and (47)

$$\hat{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \widetilde{B} = \overline{B}\hat{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\overline{K} = \begin{bmatrix} -a_{n_1} + e_3 \\ -a_{n_2} - d \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & -1 & 0 \\ -d_0 & -d_1 & 4-d_2 & 8-d_3 & 5-d_4 \end{bmatrix}$$

and

$$K = \hat{B}\overline{K}T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & -1 & 0 \\ -d_0 & -d_1 & 4-d_2 & 8-d_3 & 5-d_4 \end{bmatrix} =$$

$$= \begin{bmatrix} 1+d_0 & 2+d_1 & d_2-3 & d_3-9 & d_4-5 \\ -d_0 & -d_1 & 4-d_2 & 8-d_3 & 5-d_4 \end{bmatrix} \tag{49}$$

Using (48) and (49) we obtain the cyclic matrix

$$A_c = A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -d_0 & -d_1 & -d_2 & -d_3 & -d_4 \end{bmatrix}$$

If the pair (A,B) is uncontrollable then there exist a non-singular matrix T such that [3,4]

$$\overline{A} = TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \ \overline{B} = TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \ \begin{matrix} A_1 \in R^{r \times r}, B_1 \in R^{r \times m} \\ A_3 \in R^{(n-r)\times(n-r)} \end{matrix} \quad (50)$$

where the pair (A_1, b_1) is controllable and it has the canonical form (41) and $r = rank[A, AB, ..., A^{n-1}B] < n$.

Theorem 8. Let the pair (A,B) be uncontrollable and let the matrix A be not cyclic. Then there exists a feedback gain matrix K such that $A_c = A + BK$ is cyclic if and only if the submatrix A_3 is cyclic.

The proof is similar to the proof of Theorem 6.

6. Concluding remarks

It has been show that every second order nonzero minor of the polynomial matrix P_A of (8) is divisible (with zero remainder) by the polynomial d if and only if the characteristic polynomial $\varphi(s)$ is equal to the minimal polynomial $\Psi(s)$ of A. If the transfer matrix T has the form (3) then every second order nonzero minor of the polynomial P is divisible by d if and only if $q = d$ (q is the McMillan polynomial of T). If the pair (A,b) of single-input system is controllable then the closed-loop matrix A_c is cyclic if and only if A is cyclic. If the pair (A,B) of m-input system is controllable and A is not cyclic then there exists a feedback gain matrix K such that $A_c = A + BK$ is cyclic. If the pair (A,B) is uncontrollable and A is not cyclic then there exists a feedback gain matrix K such that $A_c = A + BK$ is cyclic if and only if the submatrix A_3 of (50) of the uncontrollable part of the system is cyclic.

The considerations with slight modifications are also valid for discrete-time linear systems. An extension of there considerations for singular linear systems will be presented in a next paper. An open problem is an extension of there considerations for standard and singular 2D linear systems [3].

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