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SOME RECENT DEVELOPMENTS IN POSITIVE AND COMPARTMENTAL SYSTEMS

Abstract: *Notions of the externally and internally positive standard and singular discrete-time and continuous-time linear systems are introduced. Necessary and sufficient conditions for the external and internal positivity of linear systems are given. It is shown that the reachability and controllability of the internally positive linear systems are not invariant under the state-feedbacks. By suitable choice of the state-feedbacks an unreachable internally positive linear systems can be made reachable and a controllable internally positive system can be made uncontrollable. The basic properties of continuous-time and discrete-time linear compartmental systems are derived and the relationships between the compartmental and positive systems are established. The realization problem for compartmental systems is formulated and partly solved.*

1. INTRODUCTION

In the last decade a dynamic development in positive and compartmental systems has been observed [1-19, 21-24]. Roughly speaking, positive systems are systems whose inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear system behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

The basic mathematical tools for analysis and synthesis of linear systems are linear spaces and the theory of linear operators. Positive linear systems are defined on cones and not on linear spaces. This is why the theory of positive systems is more complicated and less advanced. The theory of positive and compartmental systems has some elements in common with theories of linear and non-linear systems. Schematically the relationship between the theories of linear, non-linear, positive and compartmental systems is shown in the Fig.

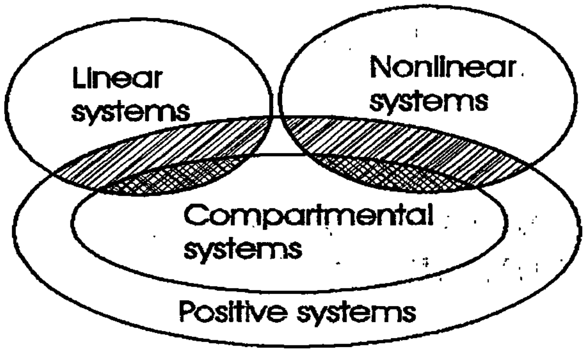

Fig.

Positive linear systems, for example, satisfy the superposition principle. Limiting the consideration of positive linear systems only to R_+^n (the first quarter of R^n) shows that the theory of positive linear systems has some elements in common with the theory of non-linear systems. An overview of the state of art in positive systems theory is given in the monographs [7, 18]. Compartmental systems is a special subclass of the positive systems.

In this paper an overview of recent developments in positive and compartmental systems will be presented. Special attention will be devoted to the relationships between positive and compartmental systems and to the realization problem for compartmental systems. Besides known results some new results will be also presented.

2. EXTERNALLY AND INTERNALLY POSITIVE SYSTEMS

2.1. Discrete-time systems

Let $R^{n\times m}$ be the set of $n\times m$ matrices with entries from the field and real numbers R and $R^n := R^{n\times 1}$. The set of $n\times m$ matrices with real non-negative entries will be denoted by $R_+^{n\times m}$ and $R_+^n := R_+^{n\times 1}$. The set of non-negative integers will be denoted by Z_+.

Consider the discrete-time linear system

(1a) $$Ex_{i+1} = Ax_i + Bu_i \quad , \quad i\in Z_+$$

(1b) $$y_i = Cx_i + Du_i$$

where $x_i\in R^n$, $u_i\in R^m$ and $y_i\in R^p$ are the state, input and output vectors and $E,A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{p\times n}$, $D\in R^{p\times m}$.

The system (1) is called singular if $\det E = 0$. If $\det E \neq 0$ then premultiplying (1a) by E^{-1} we obtain the standard system

(2a) $$x_{i+1} = Ax_i + Bu_i \quad , \quad i \in Z_+$$

(2b) $$y_i = Cx_i + Du_i$$

For the singular system (1) it is assumed that

(3) $\det[Ez - A] \neq 0$ for some $z \in \mathbf{C}$ (the field of complex numbers)

Definition 1. The standard system (2) is called externally positive if for $x_0 = 0$ and every $u_i \in R_+^m$, $i \in Z_+$ we have $y_i \in R_+^p$ for $i \in Z_+$.

Theorem 1. [18] The standard system (2) is externally positive if and only if its impulse response matrix

(4) $$g_i = \begin{cases} CA^iB & for\ i > 0 \\ D & for\ i = 0 \end{cases}$$

is non-negative $g_i \in R_+^{p \times m}$ for $i \in Z_+$

Definition 2. The standard system (2) is called internally positive if for every $x_0 \in R_+^n$ and all inputs $u_i \in R_+^m$, $i \in Z_+$ we have $x_i \in R_+^n$ and $y_i \in R_+^p$ for $i \in Z_+$.

Theorem 2. [18] The standard system (2) is internally positive if and only if

(5) $$A \in R_+^{n \times n}, B \in R_+^{n \times m}, C \in R_+^{p \times n}, D \in R_+^{p \times m}$$

The standard internally positive system (2) is always externally positive.

2.2. Continuous-time systems.

Consider the continuous-time linear system

(6a) $$E\dot{x} = Ax + Bu$$

(6b) $$y = Cx + Du$$

$\dot{x} = \dfrac{dx}{dt}$, $x = x(t) \in R^n$, $u = u(t) \in R^m$, $y = y(t) \in R^p$ are the state, input and output vectors, and $E, A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}, D \in R^{p \times m}$.

The system (6) is called singular if $\det E = 0$. If $\det E \neq 0$ then premultiplying (6a) by E^{-1} we obtain the standard system

(7a) $$\dot{x} = Ax + Bu$$

(7b) $$y = Cx + Du$$

For the singular system (6) it is assumed that

(8) $$\det[Es - A] \neq 0 \text{ for some } s \in \mathbf{C}$$

Definition 3. The standard system (7) is called externally positive if for $x_0 = x(0) = 0$ and every $u(t) \in R_+^m$, $t \geq 0$ we have $y(t) \in R_+^p$, for $t \geq 0$.

Theorem 3. [18] The standard system (7) is externally positive if and only if its impulse response matrix

(9) $$g(t) = \begin{cases} Ce^{At}B & for\ t > 0 \\ D\delta(t) & for\ t = 0 \end{cases}$$

is non-negative $g(t) \in R_+^{p \times m}$ for $t \geq 0$, where $\delta(t)$ is the Dirac impulse.

Definition 4. The standard system (7) is called internally positive if for every $x_0 \in R_+^n$ and all inputs $u(t) \in R_+^m$, $t \geq 0$ we have $x(t) \in R_+^n$ and $y(t) \in R_+^p$ for $t \geq 0$.

Theorem 4. [18] The standard system (7) is internally positive if and only if A is a Metzler matrix (all off-diagonal entries are non-negative) and $B \in R_+^{n \times m}$, $C \in R_+^{p \times n}, D \in R_+^{p \times m}$

The standard internally positive system (7) is always externally positive. The standard internally positive system (2) and (7) will be shortly called positive.

2.3. Reachability and controllability of positive 1D systems without and with feedbacks.

Definition 5. The positive system (2) is called h-step reachable if for every $x_f \in R_+^n$ (and $x_0 = 0$) there exists a input sequence $u_i \in R_+^m$, $i = 0,1,\ldots,h-1$ such that $x_h = x_f$.

Definition 6. The positive system (2) is called reachable if for every $x_f \in R_+^n$ (and $x_0 = 0$) there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,\ldots,h-1$ such that $x_h = x_f$.

Definition 7. The positive system (2) is called controllable if for every nonzero $x_f, x_0 \in R_+^n$ there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,\ldots,h-1$ such that $x_h = x_f$.

Definition 8. The positive system (2) is called controllable to zero if for every $x_0 \in R_+^n$ there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,\ldots,h-1$ such that $x_h = 0$.

Theorem 5. [6,18] The positive system (2) is n-step reachable if and only if:

i. rank $R_n = n$
ii. there exists a nonsingular matrix $\overline{R}_n$ consisting of n columns of R_n such that $R_n^{-1} \in R_+^{n \times n}$ or equivalently R_n has n linearly independent columns each containing only one positive entry

where

(10) $$R_n := \left[B, AB, \ldots, A^{n-1}B\right] \in R_+^{n\times nm}$$

If the positive system (1) is reachable then it is always n-step reachable [5,6]. □

Theorem 6. [6,18] The positive system (2) is controllable if and only if:

i. the matrix R_n has n linearly independent columns each containing only one positive entry.
ii. the spectral radius $\rho(A)$ of A is $\rho(A) < 1$ if the transfer from x_0 to x_f is allowed in an infinite number of steps and $\rho(A) = 0$ if the transfer from x_0 to x_f is required in a finite number of steps. □

Let us assume that for $m=1$ the matrices A and B of (2) have the canonical form

(11) $$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \in R_+^{n\times n},\quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R_+^n$$

It is easy to see that for (11)

(12) $$rank\left[B, AB, \ldots, A^{n-1}B\right] = n$$

but the condition ii) of theorem 5 is not satisfied if at least one $a_i \neq 0$ for $i = o, 1, \ldots, n-1$. In this case the positive system (2) with (11) is not n-step reachable.
Consider the system (2) with state-feedback

(13) $$u_i = v_i + Kx_i$$

where $K \in R^{1\times n}$ and v_i is the new input.
Substitution of (13) into (2) yields

(14) $$x_{i+1} = A_c x_i + Bv_i,\quad i \in Z_+$$

where

(15) $$A_c = A + BK$$

For (11) and

(16) $$K = \left[a_0, a_1, \ldots, a_{n-1}\right]$$

the matrix (15) has the form

(17) $$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\left[a_0, a_1, \ldots, a_{n-1}\right] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Using (17) we obtain

$$[B, A_c B, \dots, A_c^{n-1} B] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \hdashline 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Then the conditions of theorem 5 are satisfied and the closed-loop system is n-step reachable.

Therefore, the following theorem has been proved. [17,18]

Theorem 7. Let the positive system (2) with (11) be not n-step reachable. Then the closed-loop system (14) with (17) is n-step reachable if the state-feedback gain matrix K has the form (16). □

Corollary 1. The n-step reachability of positive system (2) with (11) is not invariant under the state-feedback (13).

Remark 1. It is well-known [16] that if the pair (A, B) satisfies the condition (12) then it can be transformed by linear state transformation $\overline{x}_i = Px_i$, $\det P \neq 0$ to the canonical form (11)

$$\overline{A} = PAP^{-1},\ \overline{B} = PB$$

and

$$[\overline{B}, \overline{A}\overline{B}, \dots, \overline{A}^{n-1}\overline{B}] = P[B, AB, \dots, A^{n-1}B]$$

Note that the conditions of theorem 5 are satisfied if and only if P is a monomial matrix (in each row and column has only one positive entry and the remaining entries are zero).

Consider the single-input system (2) with matrices A, B in the canonical form (11). In a similar way as in the reachability case it can be shown that the condition i) of theorem 6 is not satisfied if at least one of the coefficients $a_i \neq 0$ for $i = 0, 1, \dots, n-1$. In this case the positive system (2) with (11) is not controllable. The closed-loop system matrix (15) with (11) and state-feedback gain matrix (16) has the form (17). Note that the matrix (17) has all zero eigenvalues and its spectral radius $\rho(A_c) = 0$.

Therefore, the following theorem has been proved.

Theorem 8. Let the positive system (2) with (11) be not controllable. Then the closed-loop system (14) with (17) is controllable in a finite number of steps if the state-feedback gain matrix K has the form (16). □

The considerations can be extended with some modifications for continuous-time positive linear systems. [18]

2.3. Singular linear systems

Consider the singular discrete-time system (1) with $m = p = 1$ and

(18) $$E = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{n\times n}, \quad A = \begin{bmatrix} 0 \;\vdots\; I_{n-1} \\ \cdots\cdots \\ a \end{bmatrix} \in R^{n\times n}$$

$$a = [a_0 \;\; a_1 \cdots a_{r-1} \;\; -1 \;\; 0 \cdots 0], B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^n, C = [b_0 \;\; b_1 \cdots b_{n-1}] \in R^{1\times n}, \; D = 0$$

If (3) holds then

$$[Ez - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)}$$

where μ is the nilpotence index and Φ_i are the fundamental matrices satisfying the relation

$$E\Phi_i - A\Phi_{i-1} = \Phi_i E - \Phi_{i-1} A = \begin{cases} I \text{ (the identity matrix) for } i = 0 \\ 0 \text{ (the zero matrix) for } i \neq 0 \end{cases}$$

Theorem 9. If the matrices E, A, B, C have the canonical form (18) and

(19) $$a_i \geq 0, \; i = 0,1,\dots,r-1, b_j \geq 0, j = 0,1,\dots,n-1 \quad (n > r)$$

then

(20) $$\Phi_k B \in R_+^n \text{ for } k = -\mu, 1-\mu, \dots$$

(21) $$\Phi_i \in R_+^{n\times n} \text{ for } i \in Z_+$$

(22) $$g_j \in R_+^{p\times m} \text{ for } j = 1-\mu, 2-\mu, \dots$$

The proof is given in [11]

Theorem 10. The singular system (1) with (18) is externally and internally positive if (19) hold.

The proof follows from the relations (20)-(22).

The considerations with some modifications can be extended for continuous-time singular systems (6) [14].

3. COMPARTMENTAL SYSTEMS

3.1. Continuous-time systems

The compartmental systems consist of a finite number of subsystems called compartments [18]. Consider a compartmental system consisting of n continuous-time compartments.

Denote by $x_i=x_i(t)$ ($i=1,2,...,n$) the amount of a material of the ith compartment. Let $F_{ij}\geq 0$ be the output flow of the material from the jth to the ith compartment ($i\neq j$) and F_{0i} be the output flow of the material from the ith compartment to the environment. Let $u_i=u_i(t)$ be the input flow of the material to the ith compartment from the environment. It is assumed that the input material is mixed immediately with material being in the compartment. From the balance of material of the ith compartment we have the following differential equation

(23) $$\dot{x}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(F_{ij} - F_{ji}\right) + u_i - F_{0i} \quad \text{for } i=1,2,...,n$$

It is assumed that the flow F_{ij} depends linearly on x_j, i.e.

(24) $$F_{ij}=f_{ij}x_j \quad \text{for } i=0,1,...,n;\, j=1,2,...,n$$

where f_{ij} is a coefficient depending, in the general case, on x_j and the time instant t. The system is linear if f_{ij} is independent of x_j and it is additionally time-invariant if f_{ij} is independent of t.

Using (23) for $i=1,2,...,n$ and (24) we obtain the state equation of the compartmental system

(25) $$\dot{x} = Fx + Bu$$

where

(26) $$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},\ F = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix},\ f_{ii} := -f_{0i} - \sum_{\substack{j=1 \\ j\neq i}}^{n} f_{ji},\ B = I_n$$

Note that in every time instant the output flow of a compartment cannot be greater than the whole mass of material being inside the compartment, i.e.

(27) $$\sum_{i=1}^{n} f_{ij} \leq 0 \text{ and } f_{ij}\geq 0 \text{ for } i\neq j\ (i,j=1,2,...,n)$$

Definition 9. The matrix F satisfying the conditions (27) is called the compartmental matrix of the continuous-time system.

The first condition (27) says that the sum of entries of every column of the matrix F is not positive. The compartmental matrix is a particular case of the Metzler matrix, since $f_{ij}\geq 0$ for $i\neq j$.

The output equation has the form

(28) $$y=Cx$$

where $C\in R_+^{p\times n}$.

Therefore, the continuous-time compartmental systems are a particular case of the internally positive systems.

3.2. Discrete-time systems

Consider a compartmental system consisting of n discrete-time compartments.

In discrete-time compartmental systems the flows of materials are considered only in discrete time instants $t_1, t_2, \ldots$. It is assumed that the neighbouring time instants are shifted from each other by the unit of time, i. e. $t_{k+1}=t_k+1$.

Let $x_i(k)$, $i=1,2,\ldots,n$ be the amount of a material in the ith compartment at the time instant k. Denote by $G_{ij}(k)$ the output flow of the material from the jth to the ith compartment between the kth and $(k+1)$th time instant and by $G_{0i}(k)$ the output flow of material from the ith $(i=1,2,\ldots,n)$ compartment to the environment. It is assumed that $G_{ij}(k)$ depends linearly on $x_j(k)$, i.e.

(29) $$G_{ij}(k)=g_{ij}x_j(k) \text{ for } i=0,1,\ldots,n;\ j=1,2,\ldots,n$$

where g_{ij} is a coefficient depending, in the general case, on x_j and the discrete-time instant k.

The system is linear if g_{ij} is independent of $x_j(k)$ and it is additionally time-invariant if g_{ij} is independent of k.

Let $u_i(k)$ be the input flow of the material to the ith compartment from the environment at the time instant k.

From the balance of material of the ith compartment at the time instant $k+1$ we have the following difference equation

(30) $$x_i(k+1)=\sum_{\substack{j=1\\ j\neq i}}^{n} g_{ij}x_j(k)+g_{ii}x_i(k)+u_i(k)$$

where $g_{ii}x_i(k)$ is the amount of material in the i-th compartment at the time instant k, i.e.

(31) $$g_{ii}x_i(k)=x_i(k)-g_{0i}x_i(k)-\sum_{\substack{j=1\\ j\neq i}}^{n} g_{ji}x_i(k)=\left(1-g_{0i}-\sum_{\substack{j=1\\ j\neq i}}^{n} g_{ji}\right)x_i(k)$$

From (31) we have

(32) $$g_{ii}=1-g_{0i}-\sum_{\substack{j=1\\ j\neq i}}^{n} g_{ji},\ i=1,2,\ldots,n$$

Note that if $u_j(k)=0$ then the output flow of material at the time instant $k+1$ of the jth compartment cannot be greater than the whole mass of material being inside the compartment at the time instant k, i. e.

(33) $$\sum_{i=1}^{n} g_{ij}\le 1 \quad \text{for } j=1,2,\ldots,n \text{ and } g_{ij}\ge 0 \text{ for } i\neq j$$

Definition 10. The matrix $G \in R_+^{n\times n}$ satisfying the conditions (33) is called the compartmental matrix of the discrete-time system.
The first condition (33) says that the sum of entries of every column of the matrix G is not greater 1.
Using (30) for $i=1,2,...,n$ we obtain the state equation of the compartmental system

(34a) $$x(k+1)=Gx(k)+Bu(k)$$
where
$$x(k)=\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix},\ u(k)=\begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_n(k) \end{bmatrix},\ G=\begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}, B=I_n$$

The output equation of the compartmental system has the form

(34b) $$y(k)=Cx(k)$$
where $C \in R_+^{p\times n}$.
The discrete-time compartmental systems are particular cases of the internally positive systems.

4. RELATIONSHIP BETWEEN POSITIVE AND COMPARTMENTAL SYSTEMS

Let M be the set of Metzler matrices $A=[a_{ij}]\in R^{n\times n}$ satisfying the condition

(35) $$a_{ij}\geq 0 \text{ for } i\neq j;\ i,j=1,...,n\ .$$

and $\mathbf{C}$ be the set of compartmental matrices $A=[a_{ij}]\in R^{n\times n}$ satisfying the conditions (35) and

(36) $$\sum_{i=1}^{n} a_{ij}\leq 0 \quad for \quad j=1,...,n$$

A diagonal matrix $D=diag[d_1,d_2,...,d_n]$ is called positive if $d_i>0$ for $i=1,...,n$.

The matrix $A\in R^{n\times n}$ of the system (7) is asymptotically stable if and only if all its eigenvalues have negative real, parts [18].

Theorem 11. Let $A=[a_{ij}]\in R^{n\times n}$ be asymptotically stable. Then $A\in M$ if and only if there exists a positive diagonal matrix D such that $DAD^{-1}\in \mathbf{C}$.
Proof. By assumption the matrix A is asymptotically stable and therefore it is non-singular. To prove that if $A\in M$ then there exists a positive diagonal matrix D such that $DAD^{-1}\in \mathbf{C}$ let us define

(37) $$D = diag\{-[1,1,...,1]A^{-1}\} = diag[d_1, d_2, ..., d_n]$$

It is well-known [18] that $-A^{-1} \in R_+^{n\times n}$ and all columns of $-A^{-1}$ are nonzero. Therefore from (37) we have $d_i > 0$ for $i = 1,...,n$ and the matrix D defined by (37) is positive diagonal. Using (37) we obtain

(38) $$DAD^{-1} = \begin{bmatrix} a_{11} & a_{12}\frac{d_1}{d_2} & \cdots & a_{1n}\frac{d_1}{d_n} \\ a_{21}\frac{d_2}{d_1} & a_{22} & \cdots & a_{2n}\frac{d_2}{d_n} \\ \dots & \dots & \dots & \dots \\ a_{n1}\frac{d_n}{d_1} & a_{n2}\frac{d_n}{d_2} & \cdots & a_{nn} \end{bmatrix}.$$

The matrix (38) satisfies the condition (36) and it belongs to the set **C** since

$$[1,1,...,1]DAD^{-1} = [d_1, d_2, ..., d_n]AD^{-1} = -[1,1,...,1]A^{-1}AD^{-1} = -[1,1,...,1]D^{-1} = = -\left[\frac{1}{d_1}, \frac{1}{d_2}, ..., \frac{1}{d_n}\right].$$

From definitions of the sets M and C it follows that if $DAD^{-1} \in \mathbf{C}$ then $A \in M$. □

From Theorem 11 we have the following important corollary [3].

Corollary 2. A positive asymptotically stable system (7) is diagonally equivalent to a suitable compartmental system.

The above considerations can be extended for discrete-time systems as follows. Let $\mathbf{C}_d$ be the set of matrices $A = [a_{ij}] \in R_+^{n\times n}$ satisfying the conditions

(39) $$\sum_{i=1}^{n} a_{ij} \le 1 \text{ for } j = 1,...,n$$

The discrete-time system (2) is asymptotically stable if and only if all eigenvalues of its matrix A have moduli less 1 [18].

Theorem 12. Let $A = [a_{ij}] \in R^{n\times n}$ be asymptotically. Then $A \in R_+^{n\times n}$ if and only if there exists a positive diagonal matrix D_d such that $D_d A D_d^{-1} \in \mathbf{C}_d$.

Proof. By assumption the matrix A is asymptotically stable and therefore the matrix $[I - A]$ is non-singular. Let use define

(40) $$D_d = diag\{[1,1,...,1][I-A]^{-1}\} = diag[\bar{d}_1, \bar{d}_2, ..., \bar{d}_n]$$

It is well-known [18] that $[I-A]^{-1} \in R_+^{n\times n}$ and all columns of $[I-A]^{-1}$ are nonzero. Hence the matrix (40) is positive diagonal.

The matrix A satisfies the conditions (39) if and only if

$$[1,1,\dots,1][I-A]\geq[0,0,\dots,0]$$

and the condition $D_d A D_d^{-1}\in \mathbf{C}_d$ is equivalent to the condition $[1,1,\dots,1]D_d[I-A]D_d^{-1}\geq[0,0,\dots,0]$. Using (40) we may write

$$[1,1,\dots,1]D_d[I-A]D_d^{-1}=[\bar{d}_1,\bar{d}_2,\dots,\bar{d}_n][I-A]D_d^{-1}=$$

$$=[1,1,\dots,1][I-A]^{-1}[I-D]D_d^{-1}=[1,1,\dots,1]D_d^{-1}=\left[\frac{1}{\bar{d}_1},\frac{1}{\bar{d}_2},\dots,\frac{1}{\bar{d}_n}\right]>[0,0,\dots,0]$$

Therefore, the matrix $D_d A D_d^{-1}$ belongs to the set $\mathbf{C}_d$. □

From Theorem 12 we have the following corollary.

Corollary 3. A positive asymptotically stable system (2) is diagonally equivalent to a suitable compartmental system.

5. REALIZATION PROBLEM OF POSITIVE AND COMPARTMENTAL SYSTEMS

5.1. Positive systems

The transfer matrix of the discrete-time system (2) is given by

(41) $$T(z)=C[Iz-A]^{-1}B+D\in R^{p\times m}(z)$$

where $R^{p\times m}(z)$ is the set of proper rational matrices with real coefficients.

For the given matrices A, B, C, D there exists exactly one transfer matrix (41). On the contrary, for a given matrix $T(z)\in R^{p\times m}(z)$ there exist many different matrices A, B, C and D even of different dimensions that satisfy the equality (41).

Definition 11. Matrices A,B,C and D satisfying the equality (41) are called a realisation of a given transfer function matrix $T(z)$. □

Definition 12. A realisation (A, B, C, D) is called minimal if the matrix A has minimal dimension among all realisations of $T(z)$. □

The positive realisation problem of discrete-time systems can be formulated as follows. Given a proper rational transfer matrix $T(z)\in R^{p\times m}(z)$, find a positive realisation $A\in R_+^{n\times n}$, $B\in R_+^{n\times m}$, $C\in R_+^{p\times n}$, $D\in R_+^{p\times m}$ of $T(z)$.

For a given matrix $T(z)\in R^{p\times m}(z)$ there always exists a realisation $A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{p\times n}$, $D\in R^{p\times m}$ but there does not always exist a positive realisation. For example, for the transfer function $T(z)=\frac{b}{z+a}$, $(a>0)$, there exists a realisation of the form $A=[-a]$, $B=[b], C=[1], D=[0]$, but it is not positive if $a>0$. The problem arises of under which conditions a given matrix $T(z)$ has a positive realisation.

In a similar way the realization problem can be stated for continuous-time linear systems (7). In this case the condition $A \in R_+^{n \times n}$ should be replaced by the condition $A \in M$ (the set of Metzler matrices).

The following theorems formulate the necessary and sufficient conditions for the existence of positive realizations of a given transfer function $T(p)$, where $p=z$ for the discrete-time system and $p=s$ for the continuous-time system [3,18].

Theorem 13. Let $\{F, g, h\}$ be any minimal realization of dimension n of the transfer function $T(z)$. Then there exists a positive realization of dimension $N \geq n$ of $T(z)$ if and only if there exist matrices $A \in R_+^{N \times N}, b \in R_+^{N}$, $C \in R_+^{1 \times N}$ and $P \in R_+^{n \times N}$ such that

(42) $$FP = PA,\ g = Pb, c = hP$$

The positive realization of $T(z)$ is given by $\{A, b, c\}$.

Theorem 14. Let $\{F, g, h\}$ be any minimal realization of dimension n of the transfer function $T(s)$. Then there exists a positive realization of dimension N of $T(s)$ if and only if there exists $\alpha \in R$ and matrices $A \in R_+^{N \times N}, b \in R_+^{N}$, $c \in R_+^{1 \times N}$ and $P \in R_+^{n \times N}$ such that

(43) $$(F + \alpha I)P = PA,\ g = Pb, c = hP$$

The positive realization of $T(s)$ is given by $\{A - \alpha I, b, c\}$.

5.2. Compartmental systems

Let be given a transfer function $T(s)$. Under which conditions there exists a realization $\{A, b, c\}$ of some finite dimension of the compartmental system

(44a) $$\dot{x} = Ax + bu$$

(44b) $$y = cx$$

such that $T(s) = c[Is - A]^{-1} b$.

The following theorem given the necessary and sufficient conditions for $T(s)$ to be the transfer function of a compartmental system [3].

Theorem 15. A given asymptotically stable rational function $T(s)$ is a transfer function of a compartmental system if and only if:

1) the impulse response $g(t)$ is such that $g(t) > 0$ for every $t > 0$ and $g(0) \geq 0$;
2) the pole ρ of $T(s)$ with maximal real part is negative and unique.

Proof. *Necessity*. Let $\{A, b, c\}$ be a positive realization of $T(s)$. For a sufficiently large value of $\alpha \in R$ the matrix $\overline{A} := A + \alpha I \in R_+^{n\times n}$. Then

$$g(t) = ce^{At}b = ce^{(\overline{A}-\alpha I)t}b = e^{-\alpha t}ce^{\overline{A}t}b = e^{-\alpha t}\sum_{k=0}^{\infty} c\frac{(\overline{A}t)^k}{k!}b \tag{45}$$

From (45) the condition 1) immediately follows since at least one coefficient $c\overline{A}^k b > 0$. By assumption $T(s)$ is asymptotically stable. Hence the pole ρ of $T(s)$ with maximal real part has negative real part. It can be shown [3] that ρ is real negative and unique.

Sufficiency. Since the system is asymptotically stable then by Corollary 2 without loss of generality we can consider positive systems instead of compartmental systems.

To simplify the considerations we assume that the transfer function $\overline{T}(s) = T(s + \rho)$ has pole with maximal real part is equal to zero. There exists a positive realization of $T(s)$ if and only if there exists a positive realization of $\overline{T}(s)$ [3,18]. It can be shown [3] that if the conditions 1) and 2) are satisfied then exists a positive realization $\{A, b, c\}$ of $T(s)$.□

In [3] necessary and sufficient conditions have been established for a third-order transfer function $T(z)$ with positive real poles to be that of a positive system of the same order.

Theorem 16. [3] Let $T(s)$ be a transfer function with distinct-real poles $0 > \lambda_1 > \lambda_2 > \lambda_3$ and let $\{F, g, h\}$ be any minimal realization of $T(s)$. Then $T(s)$ has a realization with three compartments if and only if the following conditions hold:

1) $h(F - \lambda_2 I)(F - \lambda_3 I)g > 0$
2) $hg \geq 0$
3) $h(F + \overline{\alpha} I)g \geq 0$

4) $h(F + \alpha I)g \geq 0$ for all α such that $-\lambda_3 \leq \alpha \leq \overline{\alpha}$

where

$$\overline{\alpha} = \frac{\lambda_1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{3}$$

The problem of positive linear observers for linear compartmental systems has been formulated and solved in [9].

6. Concluding remarks

The notions of the externally and internally positive standard and singular discrete-time and continuous-time linear systems have been introduced. Necessary and sufficient conditions for the external and internal positivity of linear systems have been established.

It has been shown that:

1. the reachability and controllability of positive linear systems are not invariant under the state – feedbacks.
2. for an unreachable (uncontrollable) positive linear system it is possible to choose a suitable state – feedback so that the closed – loop system is reachable (controllable).

The relationships between the positive and compartmental systems have been established. The realization problem for positive and compartmental systems has been formulated and partly solved.

The presented considerations can be easily extended for multi – input continuous-time and discrete-time and also for two dimensional (2D) linear systems [10,14,16,20]. An open problem is an extension of the considerations for singular 2D linear systems.

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