Prof. dr hab. inż. Tadeusz Kaczorek
Instytut Sterowania I Elektroniki Przemysłowej
Politechnika Warszawska

INFINITE EIGENVALUE ASSIGNMENT BY OUTPUT-FEEDBACKS FOR SINGULAR SYSTEMS

Abstract. The problem of infinite eigenvalue assignment by output-feedbacks is considered. Necessary and sufficient conditions for the existence of a solution to the problem are established. A procedure for computation of the output-feedback gain matrix is given and illustrated by a numerical example.

1. INTRODUCTION

It is well-known [1,8,10,6,9] that if a pair (A,B) of standard linear system $\dot{x} = Ax + Bu$ is controllable then there exist a state-feedback gain matrix K such that $\det[I_n s - A + BK] = p(s)$, where $p(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0$ is a given arbitrary n degree polynomial. By changing K we may modify arbitrarily only the coefficients $a_0, a_1, ..., a_{n-1}$ but we are not able to change the degree n of the polynomial which is determined by the matrix $I_n s$. In singular linear systems we are also able to change the degree of the closed-loop characteristic polynomials by suitable choice of the state-feedback matrix K. The problem of finding of a state-feedback matrix K such that $\det[Es - A + BK] = \alpha \neq 0$ (α is independent of s) has been considered in [7,2]. The infinite eigenvalue assignment problem by feedbacks is very important problem in design of the perfect observers [4,5,7].
In this paper the problem of infinite eigenvalue assignment by output-feedbacks is formulated and solved.
This is an extension of the method given in [7] for output feedback case. Necessary and sufficient conditions for the existence of a solution to the problem will be established and a procedure for computation of the output-feedback gain matrix will be presented.

2. PROBLEM FORMULATION

Let $R^{n\times m}$ be the set of $n\times m$ real matrices and $R^n = R^{n\times 1}$.

Consider the continuous-time linear system

$$E\dot{x} = Ax + Bu, \; y = Cx \tag{1}$$

where $\dot{x} = \frac{dx}{dt}$, $x \in R^n$, $u \in R^m$ and $y \in R^p$ are the semistate, input and output vectors and $E, A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$. The system (1) is called singular if $\det E = 0$ and it is called standard when $\det E \neq 0$.
It is assumed that $rank\, E = r < n, rank\, B = m, rank\, C = p$ and the pair (E, A) is regular, i.e.

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbf{C} \text{ (the field of complex numbers)} \tag{2}$$

Let us consider the system (1) with the output-feedback

$$u = v - Fy \tag{3}$$

where $v \in R^m$ is a new input and $F \in R^{m\times p}$ is a gain matrix.
From (1) and (3) we have

$$E\dot{x} = (A - BFC)x + Bv \tag{4}$$

<u>**Problem 1**</u>. Given matrices E, A, B, C of (1) and nonzero scalar α (independent of s). Find a $F \in R^{m\times p}$ such that

$$\det[Es - A + BFC] = \alpha \tag{5}$$

In this paper necessary and sufficient conditions for the existence of a solution to the problem will be established and a procedure for computation of F will be proposed.

3. PROBLEM SOLUTION

From the equality

$$Es - A + BFC = [Es - A, B]\begin{bmatrix} I_n \\ FC \end{bmatrix} = [I_n, BF]\begin{bmatrix} Es - A \\ C \end{bmatrix} \tag{6}$$

and (5) it follows that the problem has a solution only if

$$rank\, [Es - A, B] = n \text{ for all finite } s \in \mathbf{C} \tag{7}$$

and

$$rank \begin{bmatrix} Es - A \\ C \end{bmatrix} = n \text{ for all finite } s \in \mathbf{C} \tag{8}$$

The problem will be solved by the use of the following two steps procedure
Step 1. (Subproblem 1). Given E, A, B of (1) and a scalar α. Find a matrix $K = FC$ such that

$$\det[Es - A + BK] = \alpha \tag{9}$$

Step 2. (subproblem 2). Given C and K depending of some free parameters $k_1, k_2, \ldots, k_l$ (found in Step 1). Find desired F satisfying the equation

$$K = FC \tag{10}$$

The solution of the subproblem 1 is based on the following lemma [2, 7].

Lemma 1. If the condition (2) is satisfied then there exist orthogonal matrices U, V such that

$$U[Es-A]V = \begin{bmatrix} E_1 s - A_1 & * \\ 0 & E_0 s - A_0 \end{bmatrix}, UB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \begin{matrix} E_1, A_1 \in R^{n_1 \times n_1} \\ E_0, A_0 \in R^{n_0 \times n_0} \end{matrix}, B_1 \in R^{n_1 \times m} \quad (11a)$$

where the subsystem (E_1, A_1, B_1) is completely controllable, the pair (E_0, A_0) is regular, E_1 is upper triangular and * denotes an unimportant matrix.

Moreover the matrices E_1, A_1 and B_1 are of the forms

$$E_1 s - A_1 = \begin{bmatrix} E_{11}s - A_{11} & E_{12}s - A_{12} & \cdots & E_{1,k-1}s - A_{1,k-1} & & E_{1k}s - A_{1k} \\ -A_{21} & E_{22}s - A_{22} & \cdots & E_{2,k-1}s - A_{2,k-1} & & E_{2k}s - A_{2k} \\ 0 & -A_{32} & \cdots & E_{3,k-1}s - A_{3,k-1} & & E_{3k}s - A_{3k} \\ \hdashline 0 & 0 & \cdots & 0 & -A_{k,k-1} & E_{kk}s - A_{kk} \end{bmatrix}, \quad (11b)$$

$$B_1 = \begin{bmatrix} B_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} E_{ij}, A_{ij} \in R^{\bar{n}_i \times \bar{n}_j}, \ i, j = 1, \dots, k \\ B_{11} \in R^{\bar{n}_1 \times m}, \sum_{i=1}^{n} \bar{n}_i = n_1 \end{matrix}$$

with $B_{11}, A_{21}, \dots, A_{k,k-1}$ of full row rank and $E_{22}, \dots, E_{kk}$ nonsingular.

Remark 1. The matrix $\overline{C} = CV$ has no special form.

Theorem 1. Let the condition (2) and (7) be satisfied and let the matrices E, A, B of (1) be transformed to the forms (11). There exists a matrix K satisfying the condition (9) if and only if

i) the subsystem (E_1, A_1, B_1) is singular, i.e.

$$\det E_1 = 0 \quad (12a)$$

ii) if $n_0 > 0$ then the degree of the polynomial $\det[E_0 s - A_0]$ is zero, i.e.

$$\deg \det[E_0 s - A_0] = 0 \text{ for } n_0 > 0 \quad (12b)$$

Proof. *Necessity.* From (9) and (11a) we have

$$\det[Es - A + BK] = \det U^{-1} \det V^{-1} \det[E_1 s - A_1 + B_1 \overline{K}] \det[E_0 s - A_0] = \alpha \quad (13)$$

where $\overline{K} = KV \in R^{m \times n}$ and $\det[E_0 s - A_0] = 1$ if $n_0 = 0$.

From (13) it follows that the condition (9) holds only if the conditions (12) are satisfied.

Sufficiency. First let us consider the single-input $(m=1)$ case. In this case we have

$$E_1=\begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n_1} \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e_{n_1n_1} \end{bmatrix}, A_1=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n_1-1} & a_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1n_1} \end{bmatrix}, B_1=b_1=\begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (14)$$

where $e_{ii}\neq 0, a_{i,i-1}\neq 0$ for $i=2,\dots,n_1$ and $b_{11}\neq 0$.

The condition (12a) implies that $e_{11}=0$. Premultiplying the matrix $[E_1s-A_1,b_1]$ by orthogonal row operations matrix P_1 it is possible to make zero the entries $e_{12},e_{13},\dots,e_{1n_1}$ of E_1 since $e_{ii}\neq 0$, $i=2,\dots,n_1$. By this reduction only the entries of the first row of A_1 will be modified.

$$\overline{E}_1=P_1E_1=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e_{n_1n_1} \end{bmatrix}, \overline{A}_1=P_1A_1=\begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1,n_1-1} & \overline{a}_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1n_1} \end{bmatrix}, \overline{b}_1=P_1b_1=b_1 \quad (15)$$

Let

$$\overline{k}_1=\frac{1}{b_{11}}\left[-\overline{a}_{11},-\overline{a}_{12},\dots,-\overline{a}_{1,n_1-1},1-\overline{a}_{1n_1}\right] \quad (16)$$

Using (13), (15) and (16) we obtain

$$\det\left[\overline{E}_1s-\overline{A}_1+\overline{b}_1\overline{k}_1\right]=$$

$$=\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ -a_{21} & e_{22}s-a_{22} & \cdots & e_{2,n_1-1}s-a_{2,n_1-1} & e_{2n_1}s-a_{2n_1} \\ 0 & -a_{31} & \cdots & e_{3,n_1-1}s-a_{3,n_1-1} & e_{3n_{1a}}s-a_{3n_{1a}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_{n_1,n_1-1} & e_{n_1n_1}s-a_{n_1n_1} \end{vmatrix}=a_{21}a_{31}\cdots a_{n_1,n_1-1}=\overline{\alpha} \quad (17)$$

where $\overline{\alpha}=\alpha\det U\det V\det P_1\det[E_0s-A_0]^{-1}$.

The considerations can be easily extended for multi-input systems, $m>1$. In this case the matrix P_1 of the orthogonal row operations is chosen so that all entries of the first row of $\overline{E}_1=P_1E_1$ are zero. By this reduction only the entries of $A_{1i}, i=1,\dots,k$ and B_{11} will be modified. The modified matrices will be denoted by $\overline{A}_{1i}, i=1,\dots,k$ and $\overline{B}_{11}$.

Let

$$\overline{K}=\overline{B}_1^{-1}\left\{\left[\overline{A}_{11},\overline{A}_{12},\dots,\overline{A}_{1k}\right]+G\right\} \quad (18)$$

The matrix $G \in R^{m \times n}$ in (18) is chosen so that

$$\overline{E}_1 s - \overline{A}_1 + \overline{B}_1 \overline{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{l+1} h \\ \overline{a}_{21} & * & \cdots & * & * \\ 0 & \overline{a}_{32} & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \overline{a}_{l,l-1} & * \end{bmatrix} \quad \text{(* denotes unimportant entries)} \quad (19)$$

$h = \dfrac{\alpha(-1)^{l+1}}{\overline{a}_{21}\overline{a}_{32}\ldots\overline{a}_{l,l-1} c}$ and $c = \det U^{-1} \det V^{-1} \det P_1^{-1} \det[E_0 s - A_0]$.

Using (13), (18) and (19) it is easy to verify that

$$\det\left[Es - A + BK\right] = c \det\left[\overline{E}_1 s - \overline{A}_1 + \overline{B}_1 \overline{K}\right] = \alpha . \quad (20)$$

Remark 2. Note that for $m > 1$ some entries of the matrix G in (18) can be chosen arbitrarily. Therefore, the matrix $K = \overline{K} V^{-1}$ has a number of free parameters denoted by $k_1, k_2, \ldots, k_l$.

The free parameters will be chosen so that the equation (10) has a solution F for given C and K.

It is well-known that the equation (10) has a solution if and only if

$$rank\, C = rank \begin{bmatrix} C \\ K \end{bmatrix} \quad (21a)$$

or equivalently

$$\operatorname{Im} K^T \subset \operatorname{Im} C^T \quad (T \text{ denotes the transpose}) \quad (21b)$$

where Im denotes the image

The free parameters $k_1, k_2, \ldots, k_l$ are chosen so that (21) holds.

Therefore, the following theorem has been proved.

Theorem 2. Let the conditions (2), (7), (8) and (12) be satisfied,

The problem has a solution, i.e. there exists F satisfying (5) if and only if the free parameters $k_1, k_2, \ldots, k_l$ of K can be chosen so that the equation (10) has a solution F for given C and K.

From the condition (21) and (16) we have the following corollary.

Corollary 1. For $m = 1$ problem has a solution if and only if the row $[\overline{a}_{11}, \overline{a}_{12}, \ldots, \overline{a}_{1n_1-1}\overline{a}_{1n_1} - 1]$ is proportional to the matrix C.

Remark 3. If the order of system is not high say $n \leq 5$ the elementary row and column operations instead of the orthogonal operations can be used.

4. EXAMPLE

For the singular system (1) with

$$E=\begin{bmatrix}0&2&1&0\\0&1&-1&2\\0&0&1&-1\\0&0&0&1\end{bmatrix}, A=\begin{bmatrix}1&-1&0&1\\0&1&2&0\\0&-1&1&-1\\0&0&2&1\end{bmatrix}, B=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}, C=\begin{bmatrix}0.5&1&3&-2\\2.5&3&4&-1\end{bmatrix} \quad (22)$$

find the gain matrix $F \in R^{2\times 2}$ such that the condition (5) is satisfied for $\alpha = 1$.
In this case the pair (*E*,*A*) is regular since

$$\det[Es-A]=\begin{vmatrix}-1&2s+1&s&-1\\0&s-1&-s-2&2s\\0&1&s-1&1-s\\0&0&-2&s-1\end{vmatrix}=(3-s)(s-1)^2-(s+2)(s-1)+4s$$

The matrices (22) have already the desired forms (11) with $A_0=0, B_0=0$, $E_1=E, A_1=A, B_1=B$, $n_1=n=4, \overline{n}_1=2, \overline{n}_2=\overline{n}_3=1, m=2$ and

$$E_{11}=\begin{bmatrix}0&2\\0&1\end{bmatrix}, E_{12}=\begin{bmatrix}1\\-1\end{bmatrix}, E_{13}=\begin{bmatrix}0\\2\end{bmatrix}, E_{22}=[1], E_{23}=[-1], E_{33}=[1]$$

$$A_{11}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}, A_{12}=\begin{bmatrix}0\\2\end{bmatrix}, A_{13}=\begin{bmatrix}1\\0\end{bmatrix}, A_{21}=[0\ \ -1], A_{22}=[1],$$

$$A_{23}=[-1], A_{32}=[2], A_{33}=[1], B_{11}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Using the elementary row operations [6,7] we obtain

$$P_1=\begin{bmatrix}1&-2&-3&1\\0&1&1&-1\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

and $$\left[\overline{E}_1 s-\overline{A}_1, \overline{B}_1\right]=P_1\left[Es-A,B\right]=\begin{bmatrix}-1&0&5&-5&1&-2\\0&s&-1&2&0&1\\0&1&s-1&1-s&0&0\\0&0&-2&s-1&0&0\end{bmatrix}$$

Taking into account that in this case

$$[\overline{A}_{11}, \overline{A}_{12}, \overline{A}_{13}] = \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \overline{B}_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & k_1 & k_2 & k_3 \end{bmatrix}$$

and using (18) we obtain

$$K = \overline{K} = \overline{B}_1^{-1}\{[\overline{A}_{11}, \overline{A}_{12}, \overline{A}_{13}] + G\} = \begin{bmatrix} 2 & 2k_1 & 2k_2 - 3 & 1 + 2k_3 \\ 0.5 & k_1 & k_2 + 1 & k_3 - 2 \end{bmatrix}$$

where k_1, k_2, k_3 are free parameters.

The free parameters are chosen so that the condition

$$rank \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} = rank \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \\ 2 & 2k_1 & 2k_2 - 3 & 1 + 2k_3 \\ 0.5 & k_1 & k_2 + 1 & k_3 - 2 \end{bmatrix} \tag{23}$$

is satisfied.

The condition (23) is satisfied for $k_1 = 1, k_2 = 2, k_3 = 0$ and the equation

$$F \begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0.5 & 1 & 3 & -2 \end{bmatrix}$$

has the solution

$$F = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

It is easy to check that

$$\det[Es - A + BK] = \det P_1^{-1} \det[\overline{E}s - \overline{A} + \overline{B}K] = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0.5 & s+1 & 2 & 0 \\ 0 & 1 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix} = 1$$

5. CONCLUDING REMARKS

The problem of infinite eigenvalue assignment by output feedbacks has been formulated and solved. Necessary and sufficient conditions for the existence of a solution to the problem have been established. Two steps procedure for computation of the output-feedback gain matrix has been derived and illustrated by a numerical example. With slight modifications the considerations can be extended for singular discrete-time linear systems. An extension of the considerations for two-dimensional linear systems [6] is also possible but it is not trivial.

6. REFERENCES

[1] Dai L., Singular Control Systems, Springer Verlag, Berlin-Tokyo 1989.

[2] Delin Chu and D.W.C Ho, Infinite eigenvalue assignment for singular systems, Linear Algebra and its Applications, vol. 298, No 1, 1999, pp. 21-37.

[3] Kaczorek T., Polynomial approach to pole shifting to infinity in singular systems by feedbacks, Bull. Pol. Acad. Techn. Sci., vol. 50, No 2 , 2002, pp. 134-144.

[4] Kaczorek T., Reduced-Order Perfect and Standard Observers for Singular Continuous-Time Linear Systems, Machine Intelligence & Robotic Control, vol. 2, No. 3, 2000, pp. 93-98.

[5] Kaczorek T., Perfect functional observers of singular continuous-time linear systems, Machine Intelligence & Robotic Control, vol. 4, No 1, 2002, pp. 77-82.

[6] Kaczorek T., Linear Control Systems, vol. 1 and 2, Research Studies Press and J. Wiley, New York 1993.

[7] Kaczorek T., The relationship between infinite eigenvalue assignment for singular systems and solvability of polynomial matrix equations, Int. J. Appl. Math. and Comp. Sci., vol. 13, No 2, 2003, pp. 161-167.

[8] Kaliath T., Linear Systems, Englewood Cliffs: Prentice Hall, 1980.

[9] Kučera V., Analysis and Design of Discrete Linear Control Systems, Academia, Prague 1981.

[10] Wonham W.M., Linear Multivariable Control: A Geometric Approach, Springer, New York 1979.