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DETERMINATION OF POSITIVE REALIZATION OF SINGULAR CONTINUOUS-TIME SYSTEMS WITH DELAYS BY STATE VARIABLES DIAGRAM METHOD

Abstract. *The positive realization problem for singular continuous-time linear single-input single-output systems with delays in state and in output (or in inputs) is addressed. A new method for determination of positive singular realizations for a given improper transfer function based on the state variables diagram is proposed. The method is presented both for systems with delays in state vector and in output and for systems with delays in state vector and in input.*

Streszczenie. *W pracy podano nową metodę schematu zmiennych stanu wyznaczania dodatniej realizacji singularnych układów ciągłych z opóźnieniami o jednym wejściu i jednym wyjściu. Proponowana metoda pozwala wyznaczyć dodatnie realizacje singularne dla danej niewłaściwej transmitancji operatorowej. Metoda ta opiera się na schemacie zmiennych stanu, który wyznaczamy znając transmitancje operatorową. Metodę tę przedstawiono dla dwóch przypadków: układu z opóźnieniami w wektorze stanu i w odpowiedziach układu oraz dla układu z opóźnieniami w wektorze stanu i wymuszeniach układu. Podano procedury wyznaczania dodatnich realizacji singularnych, które zilustrowano przykładami liczbowymi.*

1. INTRODUCTION

In positive systems inputs, state variables and output take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactor, heat exchangers and distillation column, storage systems, compartmental systems, water and atmospheric pollution model. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined one cones and not on linear space. An overview of state of the art in positive systems theory is given in monographs [3, 6]. Recent developments in positive systems theory and some new results are given in [7]. Realizations problem of positive linear systems without time-delays has been considered in many papers and books [1, 3, 6].

Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [2, 12, 13].

The realization problem for positive multivariable discrete-time systems with one time-delays was formulated and solved in [5, 9] and the realization problem for positive continuous-time systems with delay was investigated in [11].

A method for computation of a positive (singular) realization of a transfer matrix for singular continuous-time linear systems with delay in state and in output has been proposed in [10]

In this paper a new method for determination of positive singular realization for a given improper transfer function based on the state variables diagram will be proposed.

2. PRELIMINARIES AND PROBLEM FORMULATION

Let $\mathbb{R}_+^{n\times m}$ be the set of $m\times n$ real matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$. The $n\times n$ identity matrix will be denoted by $\mathbf{I}_n$.

Consider the singular single-input single-output continuous-time linear system with h delays in state and q delays in output

$$\mathbf{E}\dot{x}(t)=\sum_{i=0}^{h}\mathbf{A}_i x(t-id)+\mathbf{b}u(t) \tag{1.a}$$

$$y(t)=\sum_{j=0}^{q}\mathbf{c}_j x(t-jd) \tag{1.b}$$

where $x(t)\in\mathbb{R}^n$, $u(t)$, $y(t)\in\mathbb{R}$ are the state vector and scalar input and output, respectively and $\mathbf{E}$, $\mathbf{A}_i\in\mathbb{R}^{n\times n}$, $i=0,1,\dots,h$, $\mathbf{b}\in\mathbb{R}^n$, $\mathbf{c}_j\in\mathbb{R}^{1\times n}$, $j=0,1,\dots,q$, d is the delay.

It is assumed that det $\mathbf{E}=0$ and the characteristic polynomial is nonzero, i.e.,

$$d(s,w)=\det\left[\mathbf{E}s-\mathbf{A}_0-\mathbf{A}_1 w-\dots-\mathbf{A}_h w^h\right]\neq 0,\quad w=e^{-sd} \tag{2}$$

It is also assumed that the initial condition for (1a)

$$x_0(t),\quad t\in[-hd\quad 0]$$

belongs to the set $\mathbb{X}_0$ of admissible initial conditions.

The transfer function of the system (1) is given by

$$T(s,w)=\left(\mathbf{c}_0+\mathbf{c}_1 w+\dots+\mathbf{c}_q w^q\right)\left[\mathbf{E}s-\mathbf{A}_0-\mathbf{A}_1 w-\dots-\mathbf{A}_h w^h\right]^{-1}\mathbf{b} \tag{3}$$

It is assumed that the matrices of the system (1) have the following canonical forms [10]

$$\mathbf{E} = \begin{bmatrix} \mathbf{I}_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},\ \mathbf{A}_i = \begin{bmatrix} 0 & \mathbf{I}_{n-1} \\ \multicolumn{2}{c}{\mathbf{a}_i} \end{bmatrix},$$

$$\mathbf{a}_i = \begin{bmatrix} a_{i0} & \dots & a_{ir_i} & -1 & 0 & \dots & 0 \end{bmatrix},\ i = 0,1,\dots,h \tag{4}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n,\ \mathbf{c}_j = \begin{bmatrix} c_{j1} & c_{j2} & \dots & c_{jn} \end{bmatrix},\ j = 0,1,\dots,q$$

Definition 1. [10] *The system (1) is called (internally) positive if for any initial admissible conditions* $x_0(t)\in\mathbb{R}_+^n$, $t \in [-hd\ 0]$ *and all inputs* $u(t)\in\mathbb{R}_+$ *and its derivatives* $u^{(p)}(t)\in\mathbb{R}_+$ $(p = n-r-1)$, $t \geq 0$, $x(t)\in\mathbb{R}_+^n$ *and* $y(t)\in\mathbb{R}_+$ *for* $t \geq 0$.

Theorem 1. *The system (1) with the matrices in canonical forms (4) is positive if and only if*

(i) *The entries* a_{ik} *of the matrices* $\mathbf{A}_i$, $i = 0,1,\dots,h$ *are nonnegative except* a_{0r_0} *which can be arbitrary*

(ii) *The entries* c_{jk} *of the matrices* $\mathbf{c}_j$, $j=0,1,\dots,q$ *are nonnegative*

The proof is given in [10].

Let $\mathbf{M}_n$ be the set of $n\times n$ Metzler matrices, i.e., the set of real matrices with nonnegative off diagonal entries.

Definition 2. *Matrices*

$$\mathbf{A}_0 \in \mathbf{M}_n,\ \ \mathbf{E}, \mathbf{A}_i \in \mathbb{R}_+^{n\times n},\ \ i = 0,1,\dots,h,\ \ \mathbf{b} \in \mathbb{R}^n,\ \ \mathbf{c}_j \in \mathbb{R}_+^{1\times n},\ \ j = 0,1,\dots,q \tag{5}$$

are called a positive singular realization of a given transfer function $T(s,w)$ *if they satisfy the equality (3). A realization is called minimal if the dimension* $n\times n$ *of* $\mathbf{E}$ *and* $\mathbf{A}_i$, $i=0,1,\dots,h$ *is minimal among all realizations* $T(s,w)$.

The positive singular realization problem can be stated as follows.

Given an improper transfer function of the form

$$T(s,w) = \frac{n(s,w)}{d(s,w)} \tag{6a}$$

where

$$\begin{aligned} n(s,w) &= n_m(w)s^m + \dots + n_1(w)s + n_0(w) \\ n_k(w) &= n_{kq}w^q + \dots + n_{k1}w + n_{k0},\ \ k = 0,1,\dots,m \\ d(s,w) &= s^n - d_{n-1}(w)s^{n-1} - \dots - d_1(w)s - d_0(w) \\ d_l(w) &= d_{lq}w^q + \dots + d_{l1}w + d_{l0},\ \ l = 0,1,\dots,n-1 \end{aligned} \tag{6b}$$

and $m > n$.

Find positive singular realization (5) of $T(s,w)$.

In this paper a new method based on the state variables diagram will be proposed.

3. PROBLEM SOLUTION

3.1. Systems with delays in state and in output

Multiplying the numerator and denumerator of (6a) by s^{-m} we obtain

$$T(s,w)=\frac{n_m(w)+n_{m-1}(w)s^{-1}+\ldots+n_1(w)s^{1-m}+n_0(w)s^{-m}}{s^{n-m}-d_{n-1}(w)s^{n-m-1}-\ldots-d_1(w)s^{1-m}-d_0(w)s^{-m}}=\frac{Y(s)}{U(s)} \tag{7}$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of $y(t)$ and $u(t)$.

Let us define

$$E(s)=\frac{U(s)}{s^{n-m}-d_{n-1}(w)s^{n-m-1}-\ldots-d_1(w)s^{1-m}-d_0(w)s^{-m}} \tag{8}$$

From (8) and (7) we have

$$U(s)+\left[d_0(w)s^{-m}+\ldots+d_{n-1}(w)s^{n-m-1}-s^{n-m}\right]E(s)=0 \tag{9}$$

and

$$Y(s)=\left[n_m(w)+n_{m-1}(w)s^{-1}+\ldots+n_0(w)s^{-m}\right]E(s) \tag{10}$$

Using (9) and (10) we can draw the state variable diagram shown in Fig. 1 for m=3 and n=2.

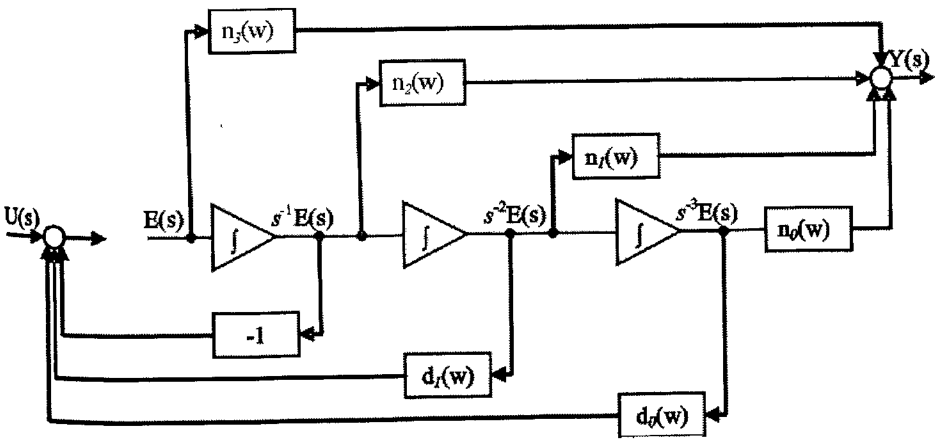

Fig. 1.

Taking into account that $d_l(w)$, l=0,1,...,n-1 and $n_k(w)$, k=0,1,...,m are polynomials in variable $w=e^{-sd}$ we obtain the complete state variables diagram shown in Fig. 2 for $n=3$ and $m=q=2$. Note that the numbers of horizontal integral elements and the vertical delay elements are equal to the maximal degrees in s and w of numerator and denominator of $T(s,w)$.

The outputs of the integral elements we choose as the state variables $x_1, x_2, \ldots, x_n$. Using the system variables diagram show in Fig. 2 we may write the following equations

$$\begin{aligned} &\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = x_3(t), \quad \dot{x}_3(t) = x_4(t), \\ &u(t) - x_3(t) + d_{10}x_2(t) + d_{11}x_2(t-d) + d_{12}x_2(t-2d) + \\ &+ d_{00}x_1(t) + d_{01}x_1(t-d) + d_{02}x_1(t-2d) = 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} y(t) = &\, n_{00}x_1(t) + n_{01}x_1(t-d) + n_{02}x_1(t-2d) + \\ &+ n_{10}x_2(t) + n_{11}x_2(t-d) + n_{12}x_2(t-2d) + \\ &+ n_{20}x_3(t) + n_{21}x_3(t-d) + n_{22}x_3(t-2d) + \\ &+ n_{30}x_4(t) + n_{31}x_4(t-d) + n_{32}x_4(t-2d) \end{aligned} \tag{12}$$

The equation (11) and (12) can be written as the following state equations

$$\mathbf{E}\dot{x}(t) = \mathbf{A}_0 x(t) + \mathbf{A}_1 x(t-d) + \mathbf{A}_2 x(t-2d) + \mathbf{b}u(t) \tag{13a}$$

$$y(t) = \mathbf{c}_0 x(t) + \mathbf{c}_1 x(t-d) + \mathbf{c}_2 x(t-2d) \tag{13b}$$

where

$$\begin{gathered} x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ d_{00} & d_{10} & -1 & 0 \end{bmatrix} \\ \mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{01} & d_{11} & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{02} & d_{12} & 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\ \mathbf{c}_0 = \begin{bmatrix} n_{00} & n_{10} & n_{20} & n_{30} \end{bmatrix}, \; \mathbf{c}_1 = \begin{bmatrix} n_{01} & n_{11} & n_{21} & n_{31} \end{bmatrix}, \\ \mathbf{c}_2 = \begin{bmatrix} n_{02} & n_{12} & n_{22} & n_{32} \end{bmatrix}. \end{gathered} \tag{14}$$

In a similar way in general case we obtain the equations (1) with matrices in canonical forms (4).

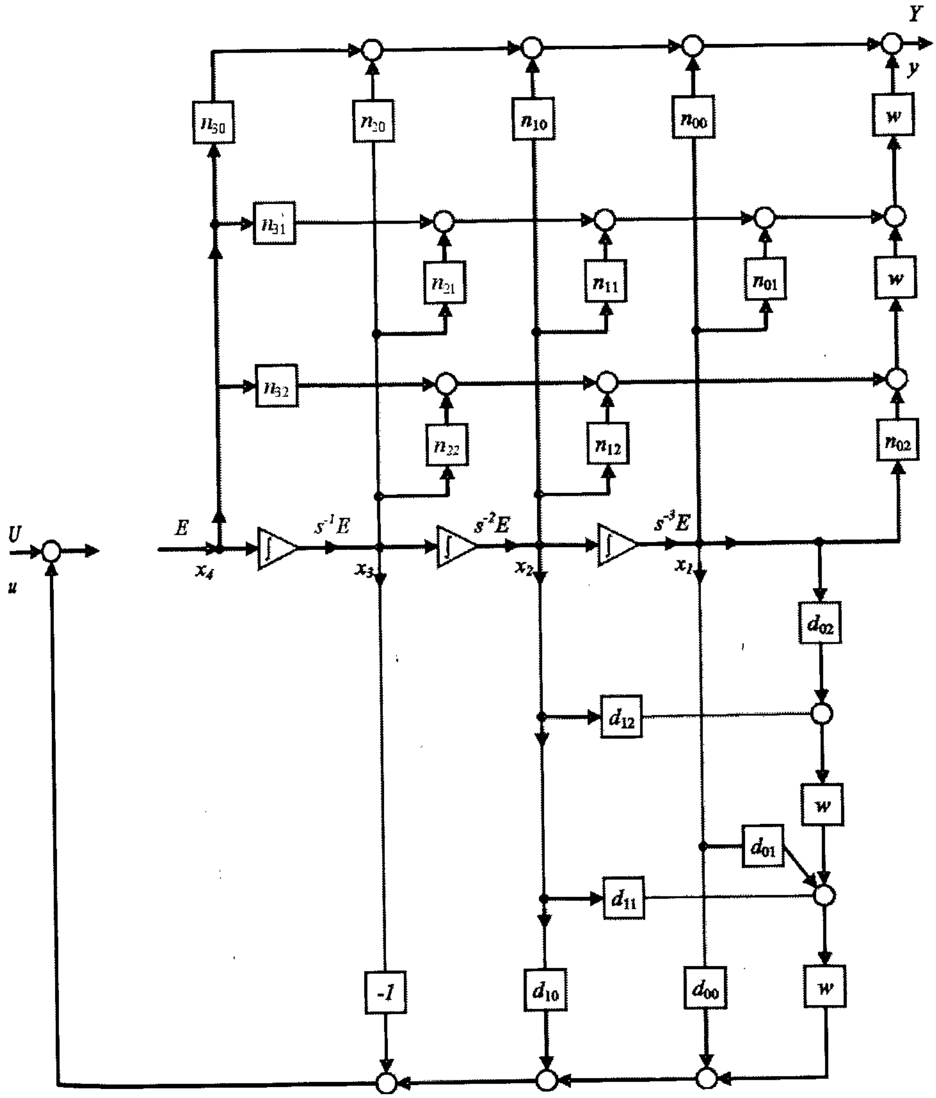

Fig. 2.

3.2. Systems with delays in state and in input

Now let us consider the singular single-input single-output continuous-time linear systems with h delays in state and q delays in input

$$\mathbf{E}\dot{x}(t) = \sum_{i=0}^{h} \mathbf{A}_i x(t - id) + \sum_{j=0}^{q} \mathbf{b}_j u(t - jd) \tag{15a}$$

$$y(t) = \mathbf{c}x(t) \tag{15b}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}$, $y(t)\in\mathbb{R}$ and E, $A_i\in\mathbb{R}^{n\times n}$, $i=0,1,\ldots,h$ have the canonical forms (4) and

$$\mathbf{b}_j=\begin{bmatrix}0\\0\\\vdots\\b_{jn}\end{bmatrix},\ j=0,1,\ldots,q,\ \ \mathbf{c}=\begin{bmatrix}c_1 & c_2 & \ldots & c_n\end{bmatrix} \tag{15c}$$

The definition of positive (internally) system of the form (15) is similar to the definition 1.

Theorem 2. *The system (15) with matrices* **E**, $\mathbf{A}_i$, $i=0,1,\ldots,h$ *in canonical forms (4) and* b_j, $j=0,1,\ldots,q$ *and* **c** *in the forms (15c) is positive if and only if*

(i) *the entries* a_{ik} *of the matrices* $\mathbf{A}_i$, $i=0,1,\ldots,h$ *are nonnegative except* a_{0r_0} *which can be arbitrary*

(ii) *the entries* b_{jn}, $j=0,1,\ldots,q$ *and* c_k, $k=1,\ldots,n$ *of the matrices* $\mathbf{b}_j$ *and* **c** *are nonnegative.*

Proof. To simplify the notation we assume $n=4$ and $h=q=2$. In this case using (14), (15c) and (11) we obtain

$$\begin{aligned}x_3(t)=&\,d_{00}x_1(t)+d_{01}x_1(t-d)+d_{02}x_1(t-2d)+\\&+d_{10}x_2(t)+d_{11}x_2(t-d)+d_{12}x_2(t-2d)+\\&+b_{04}u(t)+b_{14}u(t-d)+b_{24}u(t-2d)\end{aligned}$$

and

$$\begin{aligned}x_4(t)=\dot{x}_3(t)=&\,d_{00}\dot{x}_1(t)+d_{01}\dot{x}_1(t-d)+d_{02}\dot{x}_1(t-2d)+\\&+d_{10}\dot{x}_2(t)+d_{11}\dot{x}_2(t-d)+d_{12}\dot{x}_2(t-2d)+\\&+b_{04}\dot{u}(t)+b_{14}\dot{u}(t-d)+b_{24}\dot{u}(t-2d)=\\&=d_{00}x_2(t)+d_{01}x_2(t-d)+d_{02}x_2(t-2d)+\\&+d_{10}x_3(t)+d_{11}x_3(t-d)+d_{12}x_3(t-2d)+\\&+b_{04}\dot{u}(t)+b_{14}\dot{u}(t-d)+b_{24}\dot{u}(t-2d)\end{aligned}$$

From (11) we have

$$x_2(t)=\int_0^t x_3(t)dt+c_2 \qquad (c_2-\text{a constant})$$

$$x_1(t)=\int_0^t x_2(t)dt+c_1 \qquad (c_1-\text{a constant})$$

Therefore if the entries $\mathbf{A}_i$ and $\mathbf{b}_j$, $i,j=0,1,2$ are nonnegative then $x_k(t)\in\mathbb{R}_+$ for $t\geq 0$.

The necessity can be shown in a similar way as for standard continuous-time systems [6], [10], [11].

The transfer function of the system (15) has the form

$$T(s,w)=\mathbf{c}\left[\mathbf{E}s-\mathbf{A}_0-\mathbf{A}_1w-\ldots-\mathbf{A}_hw^h\right]^{-1}\left(\mathbf{b}_0+\mathbf{b}_1w+\ldots+\mathbf{b}_qw^q\right) \tag{16}$$

The positive singular realization problem in this case can be stated as follows.

Given an improper transfer function of the form

$$T(s,w)=\frac{n_m(w)s^m}{d(s,w)} \tag{17}$$

Where $n_m(w)$ and $d(s,w)$ are defined by (6b) and $m>n$.

Find a positive singular realization of the form

$$\mathbf{A}_0\in\mathbf{M}_n,\quad \mathbf{E},\mathbf{A}_i\in\mathbb{R}_+^{n\times n},\quad i=0,1,\ldots,h,\quad \mathbf{b}_j \text{ and } \mathbf{c} \text{ of the form (15c)} \tag{18}$$

The solution of the problem will be based on the state variables diagram.

Multiplying the numerator and denumerator of (17) by s^{-m} we obtain

$$T(s,w)=\frac{n_m(w)}{s^{n-m}-d_{n-1}(w)s^{n-m-1}-\ldots-d_1(w)s^{1-m}-d_0(w)s^{-m}}=\frac{Y(s)}{U(s)} \tag{19}$$

We define

$$E(s)=\frac{U_1(s)}{s^{n-m}-d_{n-1}(w)s^{n-m-1}-\ldots-d_1(w)s^{1-m}-d_0(w)s^{-m}} \tag{20}$$

$$U_1(s)=n_m(w)U(s)\quad\text{and}\quad Y(s)=E(s) \tag{21}$$

From (20) we have

$$U_1(s)+\left[d_{n-1}(w)s^{n-m-1}+\ldots+d_1(w)s^{1-m}+d_0(w)s^{-m}-s^{n-m}\right]E(s) \tag{22}$$

Using (22) and (21) we can draw the complete state variables diagram shown in Fig. 3 for $m=3$, $n=q=2$.

The outputs of the integral elements we choose as the state variables $x_1, x_2, \ldots, x_n$. Using the state variables diagram (Fig. 3) we may write the equations

$$\begin{aligned}&\dot{x}_1(t)=x_2(t),\quad \dot{x}_2(t)=x_3(t),\quad \dot{x}_3(t)=x_4(t)\\&d_{00}x_1(t)+d_{01}x_1(t-d)+d_{02}x_1(t-2d)+\\&+d_{10}x_2(t)+d_{11}x_2(t-d)+d_{12}x_2(t-2d)-x_3(t)+\\&+n_{30}u(t)+n_{31}u(t-d)+n_{32}u(t-2d)=0\end{aligned} \tag{23}$$

and

$$y(t)=x_4(t) \tag{24}$$

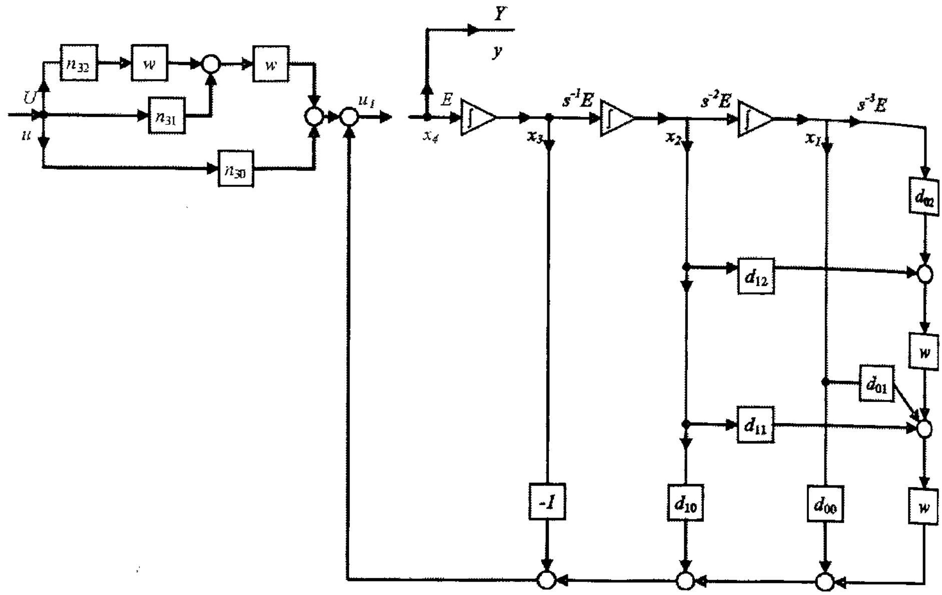

Fig. 3.

The equations (23) and (24) can be written as the following state equations

$$\mathbf{E}\dot{x}(t) = \mathbf{A}_0 x(t) + \mathbf{A}_1 x(t-d) + \mathbf{A}_2 x(t-2d) + \mathbf{b}_0 u(t) + \\ + \mathbf{b}_1 u(t-d) + \mathbf{b}_2 u(t-2d) \tag{25a}$$

$$y(t) = \mathbf{c}x(t) \tag{25b}$$

where $x(t)$, $\mathbf{E}$ and $\mathbf{A}_k$, $k = 0,1,2$ have the forms (14) and

$$\mathbf{b}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ n_{3k} \end{bmatrix}, \quad k = 0,1,2, \quad \mathbf{c} = [1 \quad 0 \quad 0 \quad 0].$$

In a similar way in general case we obtain the equations (15) with entries $\mathbf{E}$ and $\mathbf{A}_i$, i=0,1,…,h in canonical forms (4) and $\mathbf{b}_j$, j=0,1,…,q and $\mathbf{c}$ of the forms (15c).

4. CONCLUDING REMARKS

The positive realization problem for singular continuous-time single-input single-output linear systems with delays has been considered. A new method for determination of positive singular realizations for a given improper transfer function based on the state variables diagram has been proposed. The method has been presented both for systems with delays state vector and in output and for system with delays in state vector and in input. In the second case the method has been presented for systems with matrices $\mathbf{b}_j$,

j=0,1,...,q of the form (15c). An extension of the method for any $\mathbf{b}_j$, j=0,1,...,q will be presented in successive paper.

The consideration can be extended for multi-input multi-output systems in a similar way as it has been done in [10].

An extension of this method for 2D linear systems is also an open problem.

5. ACKNOWLEDGMENT

The work was supported by the State Committee for Scientific Research of Poland under grant No. 3T11A00627

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