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REACHABILITY OF FRACTIONAL POSITIVE CONTINUOUS-TIME LINEAR SYSTEMS

A new class of fractional linear continuous-time linear systems described by the state equation is introduced . The solution to the state equations is derived using the Laplace transform. Necessary and sufficient conditions are established for the internal and external positivity of the fractional systems. Sufficient conditions are given for the reachability of the fractional positive systems.

OSIĄGALNOŚĆ DODATNICH UKŁADÓW CIĄGŁYCH LINIOWYCH UŁAMKOWEGO RZĘDU

W pracy wprowadzono nową klasę układów ułamkowego rzędu opisanych równaniami stanu. Wyprowadzono rozwiązanie tych równań stosując metodę operatorową opartą na przekształceniu Laplace'a. Podano i udowodniono warunki konieczne i wystarczające wewnętrznej i zewnętrznej dodatniości układów ułamkowych. Podano również warunki wystarczające osiągalności dodatnich układów ułamkowego rzędu.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems is given in the monographs [2, 5]. An extension of positive systems are the cone systems [6, 9].

The notion of cone systems was introduced in [6]. Roughly speaking cone system is a system obtained from positive one by substitution of the positive orthants of states, inputs and outputs by suitable arbitrary cones. The realization problem for cone systems has been addressed in [6, 9].

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [12, 10, 20]. This idea has been used by engineers for modelling different process in the late 1960s [28-30]. Mathematical fundamentals of fractional calculus are given in the monographs [10, 12, 20, 13, 19]. The fractional order controllers have been developed in [18, 22]. A generalization of the Kalman filter for fractional order systems has been proposed in [26]. Some others applications of fractional order systems can be found in [1, 15-17, 3,11, 23, 24, 27, 28, 25]. In [14] a method for computation of the impulse responses from the frequency responses for the fractional standard (non-positive) discrete-time linear systems has been given. Fractional polynomials and nD systems have been investigated in [4].

In this paper a new class of fractional positive continuous-time systems described by the state equations will be introduced and the necessary and sufficient conditions for the internal and external positivity will be established.
The paper is organized as follows. In section 2 using the Caputo definition and Laplace transform the solution to the state equations of the fractional systems is derived. The necessary and sufficient conditions for the internal and external positivity of the fractional systems are established in section 3. In section 4 the reachability of the positive fractional systems is investigated. Concluding remarks are given in section 5.

To the best knowledge of the author the positive fractional continuous-time linear systems have not been considered yet.
The following notation will be used in the paper.
The set of $n \times m$ real matrices will be denoted $\Re^{n\times m}$ and $\Re^n := \Re^{n\times 1}$. The set of $m \times n$ real matrices with nonnegative entries will be denoted by $\Re_+^{m\times n}$ and $\Re_+^n := \Re_+^{n\times 1}$. A matrix A with nonnegative entries will be also denoted by $A \geq 0$. The set of nonnegative integers will be denoted by Z_+ and the $n \times n$ identity matrix by I_n.

2. CONTINUOUS-TIME FRACTIONAL LINEAR SYSTEMS AND THEIR SOLUTIONS

In this paper the following Caputo definition of the fractional derivative will be used [10, 20]

$$D^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad n-1 < \alpha \leq n \in N = \{1,2,...\} \tag{1}$$

where $\alpha \in \Re$ is the order of fractional derivative and $f^{(n)}(\tau) = \dfrac{d^n f(\tau)}{d\tau_n}$.

Consider the continuous-time fractional linear system described by the state equations

$$D^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \leq 1 \tag{2a}$$

$$y(t) = Cx(t) + Du(t) \tag{2b}$$

where $x(t) \in \Re^N$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{N\times N}$, $B \in \Re^{N\times m}$, $C \in \Re^{p\times N}$, $D \in \Re^{p\times m}$.

Theorem 1. The solution of equation (2a) is given by

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0) = x_0 \tag{3}$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \tag{4}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{5}$$

and $E_\alpha(At^\alpha)$ is the Mittage-Leffler matrix function, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the gamma function.

Proof. Using the Laplace transform (L) to (2a) and taking into account that

$$\mathcal{L}[D^\alpha x(t)] = s^\alpha X(s) - s^{\alpha-1} x_0, \quad X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t) e^{-st} dt \tag{6}$$

we obtain

$$X(s) = [I_N s^\alpha - A]^{-1}(s^{\alpha-1} x_0 + BU(s)) \tag{7}$$

where $U(s) = \mathcal{L}[u(t)]$.
It is easy to check that

$$[I_N s^\alpha - A]^{-1} = \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} \tag{8}$$

since

$$[I_N s^\alpha - A]\left(\sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}\right) = I_N. \tag{9}$$

Substitution of (8) into (7) yields

$$X(s) = \sum_{k=0}^{\infty} A^k s^{-(k\alpha+1)} x_0 + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} BU(s) \tag{10}$$

Using the inverse Laplace transformation ($\mathcal{L}^{-1}$) to (10) and the convolution theorem we obtain

$$\begin{aligned} x(t) = \mathcal{L}^{-1}[X(s)] &= \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}[s^{-(k\alpha+1)}] x_0 + \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}[s^{-(k+1)\alpha} BU(s)] \\ &= \Phi_0(t) x_0 + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau \end{aligned} \tag{11}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}[s^{-(k\alpha+1)}] = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}$$

$$\Phi(t) = \mathcal{L}^{-1}\{[I_N s^\alpha - A]^{-1}\} = \sum_{k=0}^{\infty} A^k \mathcal{L}^{-1}[s^{-(k+1)\alpha}] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \quad \blacksquare$$

Note that the solution (3) of (2a) for $Bu(t) = 0$ and $x_0 \neq 0$ is the same as in [28] but the second term of (3) is different.

Remark 1. From (4) and (5) for $\alpha = 1$ we have $\Phi_0(t) = \Phi(t) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(k+1)} = e^{At}$.

Remark 2. From the classical Cayley-Hamilton theorem we have
If

$$\det[I_N s^\alpha - A] = (s^\alpha)^N + a_{N-1}(s^\alpha)^{N-1} + \ldots + a_1 s^\alpha + a_0 \tag{12}$$

then

$$A^N + a_{N-1}A^{N-1} + \ldots + a_1 A + a_0 I = 0 \tag{13}$$

Example 1. Find the solution of equation (2a) for $0 < \alpha \le 1$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u(t) = 1(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{14}$$

Using (4) and (5) we obtain

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} = I_2 + \frac{At^\alpha}{\Gamma(\alpha+1)} \tag{15a}$$

$$\Phi(t) = I_2 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} \tag{15b}$$

since

$$A^k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for } k = 2,3,\ldots$$

Substitution of (15) and $u(t) = 1$ into(3) yields

$$\begin{aligned} x(t) &= \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau = \\ &= x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \int_0^t \left(\frac{B}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{AB}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1} \right) d\tau = \\ &= x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \frac{Bt^\alpha}{\Gamma(\alpha+1)} + \frac{ABt^{2\alpha}}{\Gamma(2\alpha+1)} = \begin{bmatrix} 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} \end{bmatrix} \end{aligned} \tag{16}$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

3. POSITIVITY OF CONTINUOUS-TIME FRACTIONAL SYSTEMS

Definition 1. The fractional system (2) is called the internally positive fractional system if and only if $x(t) \in \Re_+^N$ and $y(t) \in \Re_+^p$ for $t \ge 0$ for any initial conditions $x_0 \in \Re_+^N$ and all inputs $u(t) \in \Re_+^m$, $t \ge 0$.

A square real matrix $A = [a_{ij}]$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \ge 0$ for $i \ne j$ [1, 5].

Lemma 1. Let $A \in \Re^{N \times N}$ and $0 < \alpha \le 1$. Then

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \in \Re_+^{N \times N} \quad \text{for } t \ge 0 \tag{17}$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]} \in \Re_+^{N \times N} \quad \text{for } t \ge 0 \tag{18}$$

if and only if A is a Metzler matrix.

Prof. Necessity. From the expansion

$$\Phi_0(t) = I_N + \frac{A}{\Gamma(\alpha+1)} + \ldots$$

$$\Phi(t) = I_N \frac{t^{(\alpha-1)}}{\Gamma(\alpha)} + A \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \ldots$$

it follows that $\Phi_0(t) \in \Re_+^{N \times N}$ and $\Phi(t) \in \Re_+^{N \times N}$ for small $t > 0$ only if A is a Metzler matrix.

Sufficiency.
It is well-known [5] that

$$e^{At} \in \Re_+^{N \times N} \quad \text{for } t \ge 0 \tag{19}$$

if and only if A is a Metzler matrix.
Using (17) we may write

$$\Phi_0(t) - e^{At^\alpha} = \sum_{k=0}^{\infty} \left(\frac{(At^\alpha)^k}{\Gamma(k\alpha+1)} - \frac{(At^\alpha)^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{k! - \Gamma(k\alpha+1)}{\Gamma(k\alpha+1)} \frac{(At^\alpha)^k}{k!} \ge 0 \quad \text{for } t \ge 0 \tag{20}$$

since $k! \ge \Gamma(k\alpha+1)$ for $0 < \alpha \le 1$.

Thus from (20) and (19) we have $\Phi_0(t) \ge e^{At^\alpha} \ge 0$ for $t \ge 0$.

The proof for (18) is similar. ■

Theorem 2. The continuous-time fractional system (2) is internally positive if and only if the matrix A is a Metzler matrix and

$$B \in \Re_{+}^{N\times M}, \quad C \in \Re_{+}^{p\times N}, \quad D \in \Re_{+}^{p\times m} \tag{21}$$

Proof. Sufficiency. By Theorem 1 the solution of the equation (2a) has the form (3) and $x(t) \in \Re_{+}^{N}, t \geq 0$ if (18) holds and A is a Metzler matrix since $\Phi_0(t) \in \Re_{+}^{N\times N}$, $x_0 \in \Re_{+}^{m}$ and $u(t) \in \Re_{+}^{m}$ for $t \geq 0$.

Necessity. Let $u(t) = 0, t \geq 0$ and $x_0 = e_i$ (the *i*th column of the identity matrix I_N). The trajectory of the system does not leave the orthant $\Re_{+}^{N}$ only if $x^{\alpha}(0) = Ae_i \geq 0,$ what implies $a_{ij} \geq 0$ for $i \neq j.$ The matrix A has to be a Metzler matrix. For the same reason, for $x_0 = 0$ we have $x^{a}(0) = Bu(0) \geq 0$ what implies $B \in \Re_{+}^{N\times m}$, since $u(0) \in \Re_{+}^{m}$ may be arbitrary. From (2b) for $u(t) = 0$ we have $y(0) = Cx_0 \geq 0$ and $y(0) = Cx_0 \geq 0$ and $C \in \Re_{+}^{p\times N}$, since $x_0 \in \Re_{+}^{N}$ may be arbitrary. In a similar way, assuming $x_0 = 0$ we obtain $y(0) = Du(0) \geq 0$ and $D \in \Re_{+}^{p\times m}$, since $u(0) \in \Re_{+}^{m}$ may be arbitrary. ■

Definition 2. The fractional system (2) is called externally positive if and only if $y(t) \in \Re_{+}^{p}, t \geq 0$ for every input $u(t) \in \Re_{+}^{m}, t \geq 0$ and $x_0 = 0.$

The impulse response $g(t)$ of single-input single-output system is called its output for the input equal to the Dirac impulse $\delta(t)$ with zero initial conditions. Assuming successively that only one input is equal to $\delta(t)$ and the remaining inputs and initial conditions are zero we may define the impulse response matrix $g(t) \in \Re^{p\times m}$ of the system (2).
The impulse response matrix of the system (2) is given by

$$g(t) = C\Phi(t)B + D\delta(t) \quad \text{for} \quad t \geq 0 \tag{22}$$

Substitution of (3) into (2b) for $x_0 = 0$ yields

$$y(t) = \int_{0}^{t} C\Phi(t-\tau)Bu(\tau)d\tau + Du(t), \quad t \geq 0 \tag{23}$$

The formula (22) follows from (23) for $u(t) = \delta(t)$.

Theorem 3. The continuous-time fractional system (2) is externally positive if and only if its impulse response matrix (22) is nonnegative, i.e.

$$g(t) \in \Re_{+}^{p\times m} \quad \text{for} \quad t \geq 0 \tag{24}$$

Proof. The necessity of the condition (24) follows immediately from Definition 2. The output $y(t)$ of the system (2) with zero initial conditions for any input $u(t)$ is given by the formula

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau \tag{25}$$

which can be obtained by substitution of (22) into (23).
If the condition (24) is met and $u(t) \in \Re_+^m$, then from (25) we have $y(t) \in \Re_+^p$ for $t \geq 0$. ■

From (22) and (18) it follows that if A is a Metzler matrix and (21) holds then the impulse response matrix (22) is nonnegative. Therefore, we have the following two corollaries.

Corollary 1. The impulse response matrix (22) of the internally positive system (2) is nonnegative.

Corollary 2. Every continuous-time fractional internally positive system (2) is also externally positive.

4. REACHABILITY

Definition 3. The state $x_f \in R_+^N$ of the fractional system (2) is called reachable in time t_f if there exist an input $u(t) \in \Re_+^m$, $t \in [0, t_f]$ which steers the state of system (2) from zero initial state $x_0 = 0$ to the state x_f. If every state $x_f \in R_+^N$ is reachable in time t_f the system is called reachable in time t_f. If for every state $x_f \in R_+^N$ there exist a time t_f such that the state is reachable in time t_f then the system (2) is called reachable.

A real square matrix is called monomial if and only if each its row and column contains only one positive entry and the remaining entries are zero.

Theorem 4. The continuous-time fractional system (2) is reachable in time t_f if the matrix

$$R(t_f) = \int_0^t \Phi(\tau)BB^T\Phi^T(\tau)d\tau \tag{26}$$

is a monomial matrix.
The input which steers the state of the system (2) from $x_0 = 0$ to x_f is given by the formula

$$u(t) = B^T\Phi^T(t_f - t)R^{-1}(t_f)x_f \tag{27}$$

where T denotes the transpose.

Proof. If the matrix (26) is a monomial matrix then $R^{-1}(t_f) \in \Re_+^{N\times N}$ and the input defined by (27) is an nonnegative vector, i.e. $u(t) \in \Re_+^m$, $t \geq 0$. Using (3) for $x_0 = 0$, $t = t_f$, (27) and (26) we obtain

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau) BB^T \Phi^T (t_f - \tau) d\tau R^{-1}(t_f) x_f = \int_0^{t_f} \Phi(\tau) BB^T \Phi^T (\tau) d\tau R^{-1}(t_f) x_f = x_f$$

Therefore, the input (27) steers the state of the system (2) from $x_0 = 0$ to x_f. ■

Theorem 5. If $A = \mathrm{diag}[a_1, a_2, ..., a_N] \in \Re_+^{N\times N}$ and $B \in \Re_+^{N\times m}$ is a monomial matrix then the continuous-time fractional system (2) is reachable.

Proof. From (5) it follows that if the matrix A is diagonal then the matrix $\Phi(t)$ is also diagonal and the matrix $\Phi(t)B$ is monomial since the matrix B by assumption is monomial. From (26) written in the form

$$R(t_f) = \int_0^{t_f} \Phi(\tau) B [\Phi(\tau) B]^T d\tau \tag{28}$$

it follows that the matrix (28) is monomial. Thus by Theorem 3 the fractional system is reachable. ■

Example 2. We shall show that the fractional system (2) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{29}$$

is reachable.

Taking into account that

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for } k = 1,2,...$$

and using (5) we obtain

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]} = \begin{bmatrix} \Phi_1(t) & 0 \\ 0 & \Phi_2(t) \end{bmatrix} \tag{30}$$

where

$$\Phi_1(t) = \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]}, \quad \Phi_2(t) = \frac{t^{\alpha - 1}}{\Gamma(\alpha)}$$

and

$$\Phi(t)B = \begin{bmatrix} 0 & \Phi_1(t) \\ \Phi_2(t) & 0 \end{bmatrix}$$

In this case from (28) we have

$$R(t_f) = \int_0^{t_f} \Phi(\tau)B[\Phi(\tau)B]^T d\tau = \int_0^{t_f} \begin{bmatrix} \Phi_1^2(\tau) & 0 \\ 0 & \Phi_2^2(\tau) \end{bmatrix} d\tau \tag{31}$$

The matrix (31) is monomial and by Theorem 3 the fractional system is reachable.

Remark 3. It is well-known that the system

$$\dot{x} = Ax + Bu \tag{32}$$

with

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & a_{N-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

is reachable for any values of the coefficients a_i, $i = 0,1,\dots,N-1$, since the reachability matrix

$$[B, AB, \dots, A^{N-1}B] = I_N \tag{34}$$

The system (32) is also reachable as a positive system if $a_i \geq 0, i = 0,1,\dots,N-1$. The fractional system (2) with (33) even for $a_i = 0, i = 0,1,\dots,N-1$ is reachable if and only if there exist $u(t) > 0, t \in [0, t_f]$ such that the following condition is met

$$x_f = \int_0^{t_f} \begin{bmatrix} \dfrac{(t_f - \tau)^{\alpha-1}}{\Gamma(\alpha)} \\ \dfrac{(t_f - \tau)^{2\alpha-1}}{\Gamma(2\alpha)} \\ \dots\dots\dots\dots \\ \dfrac{(t_f - \tau)^{N\alpha-1}}{\Gamma(N\alpha)} \end{bmatrix} u(\tau) d\tau \tag{35}$$

The condition (35) follows from (3) for $x_0 = 0$, (34) and that for $a_i = 0, i = 0,1,\dots,N-1$, $A^k = 0$ for $k = N, N+1,\dots$ and

$$\Phi(t) = \sum_{k=0}^{N-1} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}$$

This example shows that the reachability conditions for the positive system (2) are much stronger than the conditions for positive system (32).

5. CONCLUDING REMARKS

A new class of fractional positive continuous-time systems has been introduced. The solution to the state equation describing the fractional systems has been derived using the Laplace transform (Theorem 1). The classical Cayley-Hamilton theorem has been extended for the fractional systems (Remark 2). Necessary and sufficient conditions have been established for the internal and external positivity of the fractional systems (Theorem 2 and 3). Sufficient conditions for the fractional positive systems are much stronger than for classical positive systems. The considerations have been illustrated by examples of fractional continuous-time linear systems.

The considerations presented for the reachability can be extended for controllability of the fractional continuous-time systems.

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